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# Gauged Linear Sigma Models for Noncompact Calabi-Yau Varieties

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**Abstract**

We study gauged linear sigma models for noncompact Calabi-Yau manifolds described as a line bundle on a hypersurface in a projective space. This gauge theory has a unique phase if the Fayet-Iliopoulos parameter is positive, while there exist two distinct phases if the parameter is negative. We find four massless effective theories in the infrared limit, which are related to each other under the Calabi-Yau/Landau-Ginzburg correspondence and the topology change. In the T-dual theory, on the other hand, we obtain two types of exact massless effective theories: One is the sigma model on a newly obtained Calabi-Yau geometry as a mirror dual, while the other is given by a Landau-Ginzburg theory with a negative power term, indicating $\mathcal{N}=2$ superconformal field theory on $SL(2,\mathbb{R})/U(1)$. We argue that the effective theories in the original gauged linear sigma model are exactly realized as $\mathcal{N}=2$ Liouville theories coupled to well-defined Landau-Ginzburg minimal models.

# 1 Introduction

Field theory in two-dimensional spacetime is one of the most powerful tools for analyzing dynamical phenomena in particle physics. It has been studied as a toy model of low energy effective theory including symmetry breaking mechanism. Nonlinear sigma model (NLSM) on the projective space is a typical example to investigate chiral symmetry breaking [1, 2]. String worldsheet theory is also described as a two-dimensional conformal field theory (CFT). Conformal invariance in the worldsheet theory gives a set of equations of motion of spacetime physics [3]. When we discuss string theory on curved spacetime, supersymmetric NLSMs play significant roles.

Coupled to a gauge field, two-dimensional field theory has been applied to more complicated physics. The gauge field plays a key role in the compactification of the target space via the Higgs mechanism. This theory is also useful to study mathematical problems such as Morse theory [4] and mirror symmetry [5]. Many people have constructed two-dimensional supersymmetric gauge theories in order to understand various kinds of physical and mathematical structures.

Higashijima and Nitta formulated supersymmetric NLSMs on hermitian symmetric spaces (HSSs), which are specific Kähler manifolds, starting from supersymmetric gauge theories with four supercharges [6]. By using this, we constructed Kähler metrics on complex line bundles over compact Einstein-Kähler manifolds [7, 8]. These noncompact Kähler manifolds have vanishing Ricci tensors, and hence are Calabi-Yau (CY) manifolds [9, 10]. In attempt to investigate the gauge/gravity duality in string theory [11, 12, 13] on these CY manifolds, however, it is indispensable to understand global aspects such as cohomology classes.

On the other hand, it is a well-known fact that the gauged linear sigma model (GLSM) is useful to investigate worldsheet string theories on toric varieties [14]. In this framework, we can understand some global properties of the CY manifolds. GLSM includes at least two kinds of SCFTs in the infrared (IR) limit. One is a supersymmetric NLSM on CY manifold and the other is an $\mathcal{N} = 2$ Landau-Ginzburg (LG) theory. We can read the cohomology ring of the CY manifold from the chiral ring derived from the LG superpotential [15]. Furthermore, its T-dual theory provides us with mirror descriptions of the original geometry [16, 17].

Since each HSS is constructed as a submanifold of a complex projective space or of a Grassmannian, we can, in general, construct the GLSMs for the line bundles on HSSs. Investigating LG theories in the IR limit of the GLSMs, we will be able to understand cohomology rings of the noncompact CY manifolds. Unfortunately, however, it is difficult to embed the sigma models on HSSs into the GLSMs. Thus we study the GLSM for a line bundle of homogeneous hypersurface in the projective space

whose Ricci tensor vanishes [18]. This noncompact manifold is represented as $\mathcal{O}(-N+\ell)$ bundle on $\mathbb{C}\mathbf{P}^{N-1}[\ell]$, which can be seen as a toy model of the line bundles on HSSs.

In this paper we will find the following theories in the IR limit: NLSMs on CY manifolds, orbifolded LG theories, gauged Wess-Zumino-Witten (WZW) models on coset $SL(2,\mathbb{R})/U(1)$ and Liouville theories. They appear as $\mathcal{N}=2$ SCFTs. The former two theories give unitary conformal field theory. Under the conformal invariance, the sigma model and the LG theory become appropriate SCFTs from differential geometric and algebro-geometric points of view, respectively. They often emerge when we analyze superstring theory on compact manifolds [19, 20]. The latter two theories are slightly different. These theories appear in string theory on noncompact curved spacetime such as a two-dimensional black hole [21, 22]. They are also utilized in non-critical string theory and matrix model [23, 24]. It is quite important to study all four SCFTs simultaneously when we consider string theories on noncompact CY manifolds. Thus we study the GLSM for noncompact CY manifolds including all the above four theories in the low energy limit.

This paper is organized as follows. In section 2 we study the GLSM for line bundles and discuss how massless effective theories appear in some specific limits. In this analysis we find that there exist two distinct phases in the negative FI parameter region. This phenomenon newly appears, while other well-known GLSMs do not include this. In section 3 we discuss the T-dual of the GLSM. We obtain two types of exact effective theories, the sigma models on newly constructed mirror CY geometries and the LG theories with negative power. There we discuss the exact effective theories in the original GLSM. We devote section 4 to the summary and discussions. In appendix A we introduce conventions of $\mathcal{N}=2$ supersymmetry in two-dimensional spacetime. In appendix B we review a definition of weighted projective space. Finally we briefly introduce the linear dilaton CFT and discuss an interpretation of LG superpotential with a negative power term in appendix C. This argument is useful to understand the LG theories in section 3.

## 2 Gauged linear sigma model

### 2.1 Lagrangian: review

First of all, let us briefly review of a general formulation of the GLSM [14]. In this model there appear various superfields such as a chiral superfield $\Phi_a$, a vector superfield $V$ and a twisted chiral superfield $\Sigma$, whose definitions are in appendix A. We also incorporate a complexified abelian gauge

transformation

$$\Phi_a \ \to \ \Phi'_a \ = \ \mathrm{e}^{-2iQ_a\Lambda}\,\Phi_a \ , \qquad \overline{\Phi}_a \ \to \ \overline{\Phi}'_a \ = \ \mathrm{e}^{+2iQ_a\overline{\Lambda}}\,\overline{\Phi}_a \ ,$$
$$V \ \to \ V' \ = \ V + i(\Lambda - \overline{\Lambda}) \ , \qquad \Sigma \ \to \ \Sigma' \ = \ \Sigma \ ,$$

where $Q_a$ is a $U(1)$ charge of the chiral superfield $\Phi_a$. For convenience, we restrict these charges to integers: $Q_a \in \mathbb{Z}$. The complexified gauge parameters are described by a chiral and an anti-chiral superfields $\Lambda(x,\theta,\overline{\theta})$ and $\overline{\Lambda}(x,\theta,\overline{\theta})$, respectively: $\overline{D}_{\pm}\Lambda = 0$, $D_{\pm}\overline{\Lambda} = 0$. By using superfields, we construct a supersymmetric gauge invariant Lagrangian:

$$\begin{aligned}\mathscr{L}_{\text{GLSM}} \ &= \ \int \mathrm{d}^4\theta \Big\{ - \frac{1}{e^2}\overline{\Sigma}\Sigma + \sum_a \overline{\Phi}_a\,\mathrm{e}^{2Q_aV}\,\Phi_a \Big\} \\ &\quad + \Big( \frac{1}{\sqrt{2}} \int \mathrm{d}^2\widetilde{\theta}\,\widetilde{W}(\Sigma) + c.c. \Big) + \Big( \int \mathrm{d}^2\theta\, W_{\text{GLSM}}(\Phi_a) + c.c. \Big) \ ,\end{aligned}$$

where we assume that all chiral superfields have non-zero $U(1)$ charges $Q_a \neq 0$ because a neutral chiral superfield is completely free from the system. The abelian gauge coupling constant $e$, which appears in front of the kinetic term of $\Sigma$, has mass dimension one. There exist two types of superpotentials. One is a superpotential written as $W_{\text{GLSM}}(\Phi_a)$. This is a holomorphic function of chiral superfields $\Phi_a$. The other is called a twisted superpotential $\widetilde{W}(\Sigma)$ described as

$$\widetilde{W}(\Sigma) \ = \ -\Sigma\, t \ , \quad t \ = \ r - i\theta \ ,$$

where $t$ is a complex parameter defined by the Fayet-Iliopoulos (FI) parameter $r$ and the Theta-angle $\theta$. We also refer $t$ to the (complexified) FI parameter.

We are interested in supersymmetric low energy effective theories. Thus we need to study the potential energy density $\mathcal{U}(\varphi)$ described by the scalar component fields of superfields:

$$\mathcal{U}(\varphi) \ = \ \frac{e^2}{2}\mathcal{D}^2 + \sum_a \big|F_a\big|^2 + \mathcal{U}_{\sigma}(\varphi) \ , \tag{2.1a}$$

$$\mathcal{D} \ := \ \frac{1}{e^2}D \ = \ r - \sum_a Q_a|\phi_a|^2 \ , \quad \overline{F}_a \ = \ -\frac{\partial}{\partial\phi_a}W_{\text{GLSM}}(\phi) \ , \quad \mathcal{U}_{\sigma}(\varphi) \ := \ 2|\sigma|^2 \sum_a Q_a^2|\phi_a|^2 \ . \tag{2.1b}$$

Where the scalar components of $\Phi_a$ and $\Sigma$ are expressed by $\phi_a$ and $\sigma$. We sometimes abbreviate scalar component fields of all superfields to $\varphi_a$. The functions $D$ and $F_a$ are auxiliary fields of $\Sigma$ and $\Phi_a$, respectively. We need not include fermionic components into the above functions (2.1) if we simply investigate supersymmetric vacua. The supersymmetric vacuum manifold $\mathcal{M}$ is defined by the vanishing potential energy density $\mathcal{U}(\varphi) = 0$:

$$\mathcal{M} \ := \ \Big\{ (\varphi_a) \in \mathbb{C}^n \,\Big|\, \mathcal{D} = F_a = \mathcal{U}_{\sigma} = 0 \Big\} \Big/ U(1) \ ,$$

where $n$ is the number of scalar component fields in the GLSM. The dividing $U(1)$ group indicates the abelian gauge symmetry. Since we consider $\mathcal{N} = (2,2)$ supersymmetric theories, the manifold $\mathcal{M}$ becomes a Kähler manifold [25] where the FI parameter denotes the scale of $\mathcal{M}$.

Under a generic configuration for chiral superfields $\Phi_a$ of charges $Q_a$, the FI parameter $r$ receives a renormalization via wave-function renormalizations of $\phi_a$. Thus the bare FI parameter $r_0$ is related to the renormalized one $r_R$ under the following equation:

$$r_0 \ = \ r_R + \sum_a Q_a \log \Big( \frac{\Lambda_{\mathrm{UV}}}{\mu} \Big) \, , \tag{2.2}$$

where $\Lambda_{\mathrm{UV}}$ and $\mu$ are the ultraviolet cut-off and the scale parameter, respectively. Thus we observe that the scale of $\mathcal{M}$ changes under the renormalization group (RG) flow. Studying the $\beta$-function of the FI parameter derived from (2.2), we find whether the effective theories expanded on $\mathcal{M}$ are asymptotically free or not. In particular, if we impose

$$\sum_a Q_a \ = \ 0 \, , \tag{2.3}$$

the FI parameter does not receive the renormalization. Thus there appears a non-trivial conformal field theory in the IR limit. From the geometric point of view, the sum $\sum_a Q_a$ is equal to the first Chern class $c_1(\mathcal{M})$ of the vacuum manifold $\mathcal{M}$. If the condition (2.3) is satisfied on $\mathcal{M}$, this manifold becomes a CY manifold. Thus we refer (2.3) to the "CY condition." In attempt to study CY manifolds, we impose this on the GLSM.

We usually study how massless effective theories are realized on the supersymmetric vacuum in $\mathcal{M}$. Recall that in two-dimensional field theory the "massless" modes are not well-defined because of the IR divergence in their two-point functions. The Coleman's theorem on non-existence of Nambu-Goldstone modes [26] is closely related to this difficulty. In order to avoid this problem, we assume that there exists an IR cut-off parameter. Furthermore we take the large volume limit $r \to \infty$ when we consider a NLSM whose target space is the vacuum manifold $\mathcal{M}$. In this limit the FI parameter $r$ of GLSM can be related to the coupling constant $g$ of the NLSM:

$$r \ = \ \frac{1}{g^2} \, .$$

This means that the large volume limit $r \to \infty$ is the weak coupling limit $g \to 0$.

Next we consider fluctuation fields around the vacuum. It is so complicated to analyze massless/massive fluctuation modes that we perform here a general calculation. Let us decompose scalar component fields $\varphi_a$ into three kinds of variables:

$$\varphi_a \ = \ \langle \varphi_a \rangle + \widetilde{\varphi}_a + \widehat{\varphi}_a \, , \qquad \int \mathrm{d}^2 x \, \big( \widetilde{\varphi}_a + \widehat{\varphi}_a \big) \ = \ 0 \, , \tag{2.4}$$

where $\langle\varphi_a\rangle$, $\widetilde{\varphi}_a$ and $\widehat{\varphi}_a$ mean the vacuum expectation values (VEVs), the fluctuation modes tangent and non-tangent to the vacuum manifold $\mathcal{M}$, respectively. They satisfy the following relations:

$$\mathscr{F}_\alpha(\varphi)\big|_{\text{VEV}} \ := \ \mathscr{F}_\alpha(\langle\varphi\rangle) \ \equiv \ 0 \, , \tag{2.5a}$$

$$\widetilde{\delta}\mathscr{F}_\alpha(\varphi)\big|_{\text{VEV}} \ := \ \sum_a \widetilde{\varphi}_a \frac{\partial \mathscr{F}_\alpha(\varphi)}{\partial \varphi_a}\Big|_{\varphi\equiv\langle\varphi\rangle} \ \equiv \ 0 \, , \tag{2.5b}$$

$$\widehat{\delta}\mathscr{F}_\alpha(\varphi)\big|_{\text{VEV}} \ := \ \sum_a \widehat{\varphi}_a \frac{\partial \mathscr{F}_\alpha(\varphi)}{\partial \varphi_a}\Big|_{\varphi\equiv\langle\varphi\rangle} \ \neq \ 0 \, . \tag{2.5c}$$

Note that $\mathscr{F}_\alpha(\varphi)$ are the set of functions given by (2.1b): $\mathscr{F}_\alpha = \{\mathcal{D}, F_a, \mathcal{U}_\sigma\}$. The symbols $\widetilde{\delta}$ and $\widehat{\delta}$ denote holomorphic variations with respect to the complex variables $\varphi_a$. Of course the VEVs $\langle\varphi_a\rangle$ satisfy the equation (2.5a). The equation (2.5b) provides that the first order variations of $\mathscr{F}_\alpha(\varphi)$ with respect to the fluctuation modes $\widetilde{\varphi}_a$ vanish. This is nothing but the definition that $\widetilde{\varphi}_a$ move only tangent to the vacuum manifold $\mathcal{M}$. The third equation (2.5c) means that the other fluctuation modes $\widehat{\varphi}_a$ do not propagate tangent to $\mathcal{M}$. Substituting (2.4) and (2.5) into the potential energy density $\mathcal{U}(\varphi)$ described by (2.1), we investigate the behaviors of the low energy effective theories. If the equations (2.5a) and (2.5b) furnish non-trivial relations among the fluctuation modes $\widetilde{\varphi}_a$, these modes constitute a supersymmetric NLSM whose target space is $\mathcal{M}$. However, if these equations are trivially satisfied, $\widetilde{\varphi}_a$ are free from constraints and propagate on a flat space with potential energy. Then we find that a field theory appears described by a superpotential of fluctuation fields such as a LG superpotential $W_{\text{LG}}$.

## 2.2 Field configuration and supersymmetric vacuum manifold

Now we are ready to analyze massless low energy effective theories in the GLSM for $\mathcal{O}(-N+\ell)$ bundle on $\mathbb{C}\mathbf{P}^{N-1}[\ell]$. We consider a $U(1)$ gauge theory with $N+2$ chiral superfields $\Phi_a$ of charges $Q_a$. We set the field configuration to

| chiral superfield $\Phi_a$ | $S_1$ | $\dots$ | $S_N$ | $P_1$ | $P_2$ |
|---|---|---|---|---|---|
| $U(1)$ charge $Q_a$ | 1 | $\dots$ | 1 | $-\ell$ | $-N+\ell$ |

(2.6)

In addition we introduce a superpotential $W_{\text{GLSM}}(\Phi) = P_1 \cdot G_\ell(S)$, where $G_\ell(S)$ is a function of chiral superfields $S_i$. This is a holomorphic homogeneous polynomial of degree $\ell$. Owing to the homogeneity, this polynomial has a following property:

$$\text{if} \ \ G_\ell(s) \ = \ \partial_1 G_\ell(s) \ = \ \dots \ = \ \partial_N G_\ell(s) \ = \ 0 \quad \to \quad \text{then} \ \ {}^\forall s_i \ = \ 0 \, . \tag{2.7}$$

By definition, the numbers $N$ and $\ell$ are positive integers: $\ell, N \in \mathbb{Z}_{>0}$. We assume that these two integers satisfy $1 \leq \ell \leq N-1$ and $2 \leq N$. The sum of all charges $Q_a$ vanishes (2.3) in order to obtain non-trivial SCFTs on the CY manifold.

Now we consider the potential energy density and look for supersymmetric vacua. Imposing the Wess-Zumino gauge, we write down the bosonic part of the potential energy density $\mathcal{U}(\varphi)$:

$$\mathcal{U}(\varphi) \ = \ \frac{e^2}{2}\mathcal{D}^2 + \big|G_{\ell}(s)\big|^2 + \sum_{i=1}^N \big|p_1 \partial_i G_{\ell}(s)\big|^2 + \mathcal{U}_{\sigma}(\varphi) \ , \tag{2.8a}$$

$$\mathcal{D} \ = \ r - \sum_{i=1}^N |s_i|^2 + \ell |p_1|^2 + (N-\ell)|p_2|^2 \ , \tag{2.8b}$$

$$\mathcal{U}_{\sigma}(\varphi) \ = \ 2|\sigma|^2 \Big\{ \sum_{i=1}^N |s_i|^2 + \ell^2 |p_1|^2 + (N-\ell)|p_2|^2 \Big\} \ . \tag{2.8c}$$

Imposing zero on them, we obtain the supersymmetric vacuum manifold $\mathcal{M}$. Since the Lagrangian has $\mathcal{N}=(2,2)$ supersymmetry and the single $U(1)$ gauge symmetry, the vacuum manifold becomes a Kähler quotient space:

$$\mathcal{M} \ = \ \Big\{ (\varphi_a) \in \mathbb{C}^{N+3} \, \Big| \, \mathcal{D} = G_{\ell} = p_1 \partial_i G_{\ell} = \mathcal{U}_{\sigma} = 0 \Big\} \Big/ U(1) \ , \tag{2.9}$$

In attempt to study effective theories, we choose a point on $\mathcal{M}$ as a vacuum and give VEVs of scalar component fields: $\varphi_a \equiv \langle \varphi_a \rangle$. Then we expand the fluctuation modes around the vacuum. In general, the structure of $\mathcal{M}$ is different for $r>0$, $r=0$ and $r<0$ and there appear various phases in the GLSM. The phase living in the $r>0$ region is referred to the "CY phase," and the phase in $r<0$ is called to the "orbifold phase." A singularity of the model emerges in the phase at $r=0$. Thus we sometimes call this the "singularity phase." We will treat these three cases separately. We comment that in each phase the vacuum manifold is reduced from the original $\mathcal{M}$. We often refer the reduced vacuum manifold to $\mathcal{M}_r \subset \mathcal{M}$. The VEVs of the respective phases can be set only in $\mathcal{M}_r$.

## 2.3 Calabi-Yau phase

In this subsection we analyze the CY phase $r>0$. In this phase, $\mathcal{D}=0$ requires some $s_i$ cannot be zero and therefore $\sigma$ must vanish. If we assume $p_1 \neq 0$, the equations $G_{\ell}(s) = \partial_i G_{\ell}(s) = 0$ with the condition (2.7) imply that all $s_i$ must vanish. However this is inconsistent with $\mathcal{D}=0$. Thus $p_1$ must be zero. The variable $p_2$ is free as long as the condition $\mathcal{D}=0$ is satisfied. Owing to these, the vacuum manifold $\mathcal{M}$ is reduced to $\mathcal{M}_{\rm CY}$ defined by

$$\mathcal{M}_{\rm CY} \ = \ \Big\{ (s_i; p_2) \in \mathbb{C}^{N+1} \, \Big| \, \mathcal{D} = G_{\ell}(s) = 0 \, , \ r > 0 \Big\} \Big/ U(1) \ . \tag{2.10}$$

Here we explain this manifold in detail. This is an $(N-1)$-dimensional noncompact Kähler manifold. The components $s_i$ denote the homogeneous coordinates of the complex projective space $\mathbb{C}\mathbf{P}^{N-1}$. The constraint $G_{\ell}(s)=0$ reduces $\mathbb{C}\mathbf{P}^{N-1}$ to a degree $\ell$ hypersurface expressed to $\mathbb{C}\mathbf{P}^{N-1}[\ell]$. we find that

$p_2$ is a fiber coordinate of the $\mathcal{O}(-N+\ell)$ bundle on $\mathbb{C}\mathbf{P}^{N-1}[\ell]$. Furthermore the vanishing sum of $U(1)$ charges indicates that the FI parameter $r$ is not renormalized. This is equivalent to $c_1(\mathcal{M}_{\rm CY}) = 0$. Thus we conclude that the reduced vacuum manifold $\mathcal{M}_{\rm CY}$ is nothing but a noncompact CY manifold on which a non-trivial superconformal field theory is realized.

Let us consider a low energy effective theory. We choose a vacuum and take a set of VEVs of the scalar component fields. Because ${}^{\exists}\langle s_i\rangle \neq 0$, the $U(1)$ gauge symmetry is spontaneously broken down completely. Next, we expand all fields in terms of fluctuation modes such as $\varphi_a = \langle\varphi_a\rangle + \widetilde{\varphi}_a + \widehat{\varphi}_a$. We set $\langle p_1\rangle$, $\langle\sigma\rangle$ and $\widetilde{\sigma}$ to be zero. Substituting them into the potential energy density (2.8), we obtain

$$
\begin{aligned}
\mathcal{U} \ = \ & \frac{e^2}{2}\Big\{ 2\mathrm{Re}\Big[ -\sum_{i=1}^N 2\widehat{s}_i\langle\overline{s}_i\rangle + (N-\ell)\widehat{p}_2\langle\overline{p}_2\rangle\Big] - \sum_{i=1}^N \big|\widetilde{s}_i + \widehat{s}_i\big|^2 + \ell|\widehat{p}_1|^2 + (N-\ell)\big|\widetilde{p}_2 + \widehat{p}_2\big|^2\Big\}^2 \\
& + \Big| \sum_{i=1}^N \widehat{s}_i\,\partial_i G_\ell(\langle s\rangle) + \sum_{k=2}^{\ell}\frac{1}{k!}\sum_{i_1,\cdots,i_k}(\widetilde{s}+\widehat{s})_{i_1}\cdots(\widetilde{s}+\widehat{s})_{i_k}\cdot\partial_{i_1}\cdots\partial_{i_k}G_\ell(\langle s\rangle)\Big|^2 \\
& + |\widehat{p}_1|^2\sum_{i=1}^N\Big|\partial_i G_\ell(\langle s\rangle) + \sum_{k=2}^{\ell-1}\frac{1}{k!}\sum_{j_1,\cdots,j_k}(\widetilde{s}+\widehat{s})_{j_1}\cdots(\widetilde{s}+\widehat{s})_{j_k}\cdot\partial_i\partial_{j_1}\cdots\partial_{j_k}G_\ell(\langle s\rangle)\Big|^2 \\
& + 2|\widehat{\sigma}|^2\Big\{\sum_{i=1}^N\big|\langle s_i\rangle + \widetilde{s}_i + \widehat{s}_i\big|^2 + \ell^2|\widehat{p}_1|^2 + (N-\ell)^2\big|\langle p_2\rangle + \widetilde{p}_2 + \widehat{p}_2\big|^2\Big\}\,.
\end{aligned}
$$

Fluctuation modes $\widetilde{s}_i$ and $\widetilde{p}_2$ remain massless and move only tangent to $\mathcal{M}_{\rm CY}$ because they are subject to $\widetilde{\delta}\mathcal{D}|_{\rm VEV} = \widetilde{\delta}G_\ell|_{\rm VEV} = 0$. The variation $\widetilde{\delta}(p_1\partial_i G_\ell)|_{\rm VEV} = 0$ indicates $\widetilde{p}_1 = 0$. The modes $\widehat{\sigma}$, $\widehat{p}_1$, $\widehat{s}_i$ and $\widehat{p}_2$ have mass $m^2 = \mathcal{O}(e^2 r)$. The gauge field $v_m$ also acquires mass of order $\mathcal{O}(e^2 r)$ by the Higgs mechanism. The fermionic superpartners behave in the same way as the scalar component fields because of preserving supersymmetry. In the IR limit $e\to\infty$ and the large volume limit $r\to\infty$, the massive modes decouple from the system. Thus we obtain

$$
\mathcal{N} = (2,2) \text{ supersymmetric NLSM on } \mathcal{M}_{\rm CY} \tag{2.11}
$$

as a massless effective theory. Notice that this description is only applicable in the large volume limit because the NLSM is well-defined in the weak coupling limit from the viewpoint of perturbation theory. This effective theory becomes singular if we take the limit $r\to +0$ because the decoupled massive modes becomes massless. This phenomenon also appears in the Seiberg-Witten theory [27, 28], the black hole condensation [29, 30], and so on.

Let us make a comment on the target space $\mathcal{M}_{\rm CY}$. By definition, the number $\ell$ means the degree of the vanishing polynomial $G_\ell(s) = 0$, which gives a hypersurface in the projective space $\mathbb{C}\mathbf{P}^{N-1}$. We can see that if $\ell = 1$, $G_{\ell=1}(s) = 0$ gives a linear constraint with respect to the homogeneous coordinates $s_i$ and the hypersurface $\mathbb{C}\mathbf{P}^{N-1}[\ell = 1]$ is reduced to $(N-2)$-dimensional projective space

$\mathbb{C}\mathbf{P}^{N-2}$. This reduction does not occur if $2 \le \ell \le N-1$. Here we summarize the shape of the target space $\mathcal{M}_{\rm CY}$ in Table 1:

| degree $\ell$ | vacuum manifold $\mathcal{M}_{\rm CY}$ |
|---|---|
| $\ell = 1$ | $\mathcal{O}(-N+1)$ bundle on $\mathbb{C}\mathbf{P}^{N-2}$ |
| $2 \le \ell \le N-1$ | $\mathcal{O}(-N+\ell)$ bundle on $\mathbb{C}\mathbf{P}^{N-1}[\ell]$ |

Table 1: *Classification of* $\mathcal{O}(-N+\ell)$ *bundle on* $\mathbb{C}\mathbf{P}^{N-1}[\ell]$.

Although the $\ell = 1$ case has been already analyzed in the original paper [14], the other cases $2 \le \ell \le N-1$ are the new ones which have not been analyzed.

### 2.4 Orbifold phase

Here we consider the negative FI parameter region $r < 0$. In this region the total vacuum manifold (2.9) is restricted to a subspace defined by

$$\mathcal{M}_{\rm orbifold} \ = \ \Big\{ (p_1, p_2; s_i) \in \mathbb{C}^{N+2} \, \Big| \, \mathcal{D} = G_{\ell} = p_1 \partial_i G_{\ell} = 0 \, , \ r < 0 \Big\} \Big/ U(1) \, . \qquad (2.12)$$

Since $\mathcal{D} = 0$ does not permit $p_1$ and $p_2$ to vanish simultaneously, $\sigma$ must be zero. This subspace is quite different from $\mathcal{M}_{\rm CY}$ in the CY phase. In addition, the shape of $\mathcal{M}_{\rm orbifold}$ is sensitive to the change of the degree $\ell$ because of the existence of the constraints $G_{\ell} = p_1 \partial_i G_{\ell} = 0$ and the property (2.7). Thus let us analyze $\mathcal{M}_{\rm orbifold}$ and study massless effective theories on it in the case of $3 \le \ell \le N-1$, $\ell = 2$ and $\ell = 1$, separately.

#### Effective theories of $3 \le \ell \le N-1$

Here we analyze the vacuum manifold $\mathcal{M}_{\rm orbifold}$ and massless effective theories of $3 \le \ell \le N-1$. Owing to the constraints $G_{\ell} = p_1 \partial_i G_{\ell} = 0$ and their property (2.7), the manifold $\mathcal{M}_{\rm orbifold}$ is decomposed into the following two subspaces:

$$\mathcal{M}_{\rm orbifold}\big|_{3 \le \ell \le N-1} \ = \ \mathcal{M}^1_{r<0} \cup \mathcal{M}^2_{r<0} \, , \qquad (2.13a)$$

$$\mathcal{M}^1_{r<0} \ := \ \Big\{ (p_1, p_2) \in \mathbb{C}^2 \, \Big| \, \mathcal{D} = 0 \, , \ r < 0 \Big\} \Big/ U(1) \, , \qquad (2.13b)$$

$$\mathcal{M}^2_{r<0} \ := \ \Big\{ (p_2; s_i) \in \mathbb{C}^{N+1} \, \Big| \, \mathcal{D} = G_{\ell} = 0 \, , \ r < 0 \Big\} \Big/ U(1) \, . \qquad (2.13c)$$

In the former subspace the condition (2.7) is trivially satisfied whereas in the latter subspace it is satisfied non-trivially. Both of the two subspace include a specific region $p_1 = {}^{\forall} s_i = 0$. The

subspace $\mathcal{M}^1_{r<0}$ is defined as a one-dimensional *weighted* projective space $\mathbf{W}\mathbb{C}\mathbf{P}^1_{\ell,N-\ell}$ represented by two complex fields $p_1$ and $p_2$ of $U(1)$ charges $-\ell$ and $-(N-\ell)$, respectively. The precise definition of the weighted projective space is in appendix B.

Let us choose a supersymmetric vacuum and set VEVs of all scalar fields. Then we expand all the fields around the VEVs. Expanding the potential energy density (2.8) in terms of VEVs and fluctuation modes, we obtain the following form:

$$\begin{aligned}
\mathcal{U} \ &= \ \frac{e^2}{2}\Big\{2\,\mathrm{Re}\Big[\ell\,\widehat{p}_1\langle\overline{p}_1\rangle + (N-\ell)\widehat{p}_2\langle\overline{p}_2\rangle\Big] - \sum_i\big|\widetilde{s}_i+\widehat{s}_i\big|^2 + \ell\big|\widetilde{p}_1+\widehat{p}_1\big|^2 + (N-\ell)\big|\widetilde{p}_2+\widehat{p}_2\big|^2\Big\}^2 \\
&\quad + \big|G_\ell(\widetilde{s}+\widehat{s})\big|^2 + \big|\langle p_1\rangle+\widetilde{p}_1+\widehat{p}_1\big|^2\cdot\sum_i\big|\partial_i G_\ell(\widetilde{s}+\widehat{s})\big|^2 \\
&\quad + 2|\widehat{\sigma}|^2\Big\{\sum_i\big|\widetilde{s}_i+\widehat{s}_i\big|^2 + \ell^2\big|\langle p_1\rangle+\widetilde{p}_1+\widehat{p}_1\big|^2 + (N-\ell)^2\big|\langle p_2\rangle+\widetilde{p}_2+\widehat{p}_2\big|^2\Big\}\ ,
\end{aligned}$$

where $\langle p_1\rangle$ and $\langle p_2\rangle$ are VEVs of scalar components of $P_1$ and $P_2$, respectively. They live in the weighted projective space (2.13b). Because the VEVs of $s_i$ are all zero, the $U(1)$ gauge symmetry is spontaneously broken to $\mathbb{Z}_\alpha$, where $\alpha$ is the great common number between $\ell$ and $N-\ell$: $\alpha = \mathrm{GCM}\{\ell, N-\ell\}$. This potential energy density provides that all fluctuation modes $\widetilde{s}_i$ and $\widehat{s}_i$ appear as linearly combined forms such as $\widetilde{s}_i+\widehat{s}_i$, which do not acquire any mass terms. The modes $\widetilde{p}_1$ and $\widetilde{p}_2$ remain massless and move tangent to the subspace (2.13b). The other fluctuation modes acquire mass of order $m^2 = \mathcal{O}(e^2|r|)$. Thus, in the IR limit $e\to\infty$, all the massive modes are decoupled from the system. Thus we obtain the following massless effective theory:

$$\begin{aligned}
&\mathcal{N}=(2,2) \text{ supersymmetric NLSM on } \mathbf{W}\mathbb{C}\mathbf{P}^1_{\ell,N-\ell} \\
&\text{coupled to “LG” theory with } \Big\{W_{\mathrm{LG}} \ = \ (\langle p_1\rangle + P_1)G_\ell(S)\Big\}\Big/\mathbb{Z}_\alpha\ ,
\end{aligned} \tag{2.14}$$

where $P_1$ and $S_i$ are massless chiral superfields. Note that the sigma model sector also contains the $\mathbb{Z}_\alpha$ orbifold symmetry coming from the property of $\mathbf{W}\mathbb{C}\mathbf{P}^1_{\ell,N-\ell}$. As is well known that the term $\langle p_1\rangle G_\ell(S)$ forms an ordinary LG superpotential. Thus in the IR limit we can interpret that this term is marginal and flows to the $\mathcal{N}=(2,2)$ minimal model. The second term $P_1\cdot G_\ell(S)$ is somewhat mysterious. Since this term has not any isolated singularities we might not obtain well-defined unitary CFT. This difficulty causes the noncompactness of the manifold $\mathcal{M}_{\mathrm{CY}}$ which appears in the CY phase.

There are two specific points in the subspace $\mathbf{W}\mathbb{C}\mathbf{P}^1_{\ell,N-\ell}$. One is the point $p_2=0$ and the other is $p_1=0$. In the former point the gauge symmetry is enhanced to $\mathbb{Z}_\ell$. Furthermore the mode $\widetilde{p}_1$ disappears and $\widehat{p}_2$ becomes massless, which combines with a massless fluctuation modes $\widetilde{p}_2$ linearly. This combined mode is free from any constraints. The other massless modes $\widetilde{s}_i+\widehat{s}_i$ in (2.14) remain massless and are also free from constraints. Thus in the IR limit and the large volume limit, the

massless effective theory becomes an $\mathcal{N}=(2,2)$ supersymmetric theory as

$$\Big\{\text{CFT on } \mathbb{C}^1 \ \otimes\ \text{LG theory with } W_{\text{LG}} = \langle p_1\rangle G_\ell(S)\Big\}\Big/\mathbb{Z}_\ell\ . \tag{2.15}$$

This effective theory consists of $N+1$ massless chiral superfields such as $P_2$ and $S_i$, which live in the free and the LG sectors, respectively. Since we take the IR limit, this effective theory becomes an SCFT. The LG sector flows to a well-known LG minimal model [14]. Thus the sigma model sector is also a superconformal field theory. Here we notice that we did not integrate out but just decomposed all massive modes in the above discussion because it is generally impossible to calculate the integration of them. Thus the above effective theory is merely an approximate one. If we will be able to integrate out all massive modes exactly, the obtaining effective theory will be different from the above one. In later section we will discuss the exact form of the effective theory.

Next let us consider the latter point $p_1=0$ in the space $\mathbf{W}\mathbb{C}\mathbf{P}^1_{\ell,N-\ell}$. On this point the broken gauge symmetry is partially restored to $\mathbb{Z}_{N-\ell}$. The massless fluctuation mode $\widetilde{p}_2$ becomes zero whereas the massive mode $\widehat{p}_1$ becomes massless, which combines with $\widetilde{p}_1$ being free from any constraints. Thus $P_1$ appears as a massless chiral superfield. In the IR limit we obtain the supersymmetric massless effective theory such as

$$\Big\{\text{LG theory with } W_{\text{LG}} = P_1\cdot G_\ell(S)\ \text{ on } \mathbb{C}^{N+1}\Big\}\Big/\mathbb{Z}_{N-\ell}\ , \tag{2.16}$$

which consists of $N+1$ massless chiral superfields such as $P_1$ and $S_i$. This theory is not a well-defined LG theory because the superpotential $W_{\text{LG}}$ has no isolated singularities. We interpret the defect of isolated singularities as a noncompactness of the manifold $\mathcal{M}_{\text{CY}}$ in the CY phase via CY/LG correspondence (if this correspondence is satisfied in the case of sigma models on noncompact CY manifolds.) This property prevents from calculating a chiral ring of this model in the same way as unitary LG minimal models describing compact CY manifolds [15].

Here we study massless effective theories on the subspace $\mathcal{M}^2_{r<0}$ defined in (2.13c). As mentioned before, there are non-trivial constraints in $\mathcal{M}^2_{r<0}$. Thus, as we shall see, the effective theories are also under these constraints. In the same way as discussed before, we choose one point in the subspace $\mathcal{M}^2_{r<0}$ and make all the scalar fields fluctuate around it. Then we write down the expanded potential energy density (2.8) in terms of VEVs and fluctuation modes $\langle\varphi_a\rangle$, $\widetilde{\varphi}_a$ and $\widehat{\varphi}_a$:

$$\begin{aligned}
\mathcal{U} \ &=\ \frac{e^2}{2}\Big\{2\,\mathrm{Re}\Big[-\sum_i \widehat{s}_i\langle\overline{s}_i\rangle + (N-\ell)\widehat{p}_2\langle\overline{p}_2\rangle\Big] - \sum_i \big|\widetilde{s}_i+\widehat{s}_i\big|^2 + \ell\big|\widehat{p}_1\big|^2 + (N-\ell)\big|\widetilde{p}_2+\widehat{p}_2\big|^2\Big\}^2 \\
&\quad + \Big|\sum_i \widehat{s}_i\,\partial_i G_\ell(\langle s\rangle) + \sum_{k=2}^{\ell}\frac{1}{k!}\sum_{i_1,\ldots,i_k}(\widetilde{s}+\widehat{s})_{i_1}\cdots(\widetilde{s}+\widehat{s})_{i_k}\cdot\partial_{i_1}\cdots\partial_{i_k}G_\ell(\langle s\rangle)\Big|^2
\end{aligned}$$

$$+|\widehat{p}_1|^2 \cdot \sum_i \Big| \partial_i G_\ell(\langle s \rangle) + \sum_{k=1}^{\ell-1} \frac{1}{k!} \sum_{j_1,\ldots,j_k} (\widetilde{s}+\widehat{s})_{j_1} \cdots (\widetilde{s}+\widehat{s})_{j_k} \cdot \partial_i \partial_{j_1} \cdots \partial_{j_k} G_\ell(\langle s \rangle) \Big|^2$$
$$+ 2|\widehat{\sigma}|^2 \Big\{ \sum_i \big| \langle s_i \rangle + \widetilde{s}_i + \widehat{s}_i \big|^2 + \ell^2 \big| \widehat{p}_1 \big|^2 + (N-\ell)^2 \big| \langle p_2 \rangle + \widetilde{p}_2 + \widehat{p}_2 \big|^2 \Big\} .$$

This potential energy density indicates the following: The fluctuation modes $\widehat{s}_i$, $\widehat{p}_1$ and $\widehat{p}_2$ are massive; $\widetilde{s}_i$ and $\widetilde{p}_2$ move tangent to $\mathcal{M}^2_{r<0}$. Thus, taking $e \to \infty$ and $|r| \to \infty$, we obtain

$$\mathcal{N} = (2,2) \text{ supersymmetric NLSM on } \mathcal{M}^2_{r<0} . \tag{2.17}$$

In this theory there exist $P_2$ and $S_i$ as massless chiral superfields, which move tangent to $\mathcal{M}^2_{r<0}$. Notice that in general points in $\mathcal{M}^2_{r<0}$ the $U(1)$ gauge symmetry is completely broken because of the existence of non-zero VEVs $\langle s_i \rangle$. However, taking ${}^\forall \langle s_i \rangle = \langle p_1 \rangle = 0$ and $\langle p_2 \rangle \neq 0$ in the subspace $\mathcal{M}^2_{r<0}$, we find that the gauge symmetry is partially restored to $\mathbb{Z}_{N-\ell}$.

Even though the vacuum manifolds $\mathcal{M}^1_{r<0}$ and $\mathcal{M}^2_{r<0}$ are connected on $p_1 = {}^\forall s_i = 0$, the effective theories given by (2.16) and (2.17) are quite different from each other. The reason is that while the subspace $\mathcal{M}^1_{r<0}$ is free from constraints $G_\ell = p_1 \partial_i G_\ell = 0$, in the subspace $\mathcal{M}^2_{r<0}$ these constraints are still valid on the region $p_1 = {}^\forall s_i = 0$. On account of the existence of these constraints, a phase transition occurs when the theory moves from one to the other. Thus we conclude that a new phase appears on the subspace $\mathcal{M}^2_{r<0}$, which has not been discovered in well-known GLSMs such as the models for $\mathcal{O}(-N)$ bundle on $\mathbb{C}\mathbf{P}^{N-1}$, for $\mathbb{C}\mathbf{P}^{N-1}[N]$, for resolved conifold, and so on. We refer this phase to the "3rd phase." Here we refer the phase on $\mathcal{M}^1_{r<0}$ to the orbifold phase, as usual.

### Effective theories of $\boldsymbol{\ell = 2}$

Let us consider the orbifold phase of $\ell = 2$. In the same way as the previous analysis, the constraints $G_\ell = p_1 \partial_i G_\ell = 0$ and the property (2.7) decompose the manifold $\mathcal{M}_{\text{orbifold}}$ into two subspaces:

$$\mathcal{M}_{\text{orbifold}}\big|_{\ell=2} = \mathcal{M}^1_{r<0} \cup \mathcal{M}^2_{r<0} , \tag{2.18a}$$
$$\mathcal{M}^1_{r<0} := \Big\{ (p_1, p_2) \in \mathbb{C}^2 \Big| \mathcal{D} = 0 , \ r < 0 \Big\} \Big/ U(1) \equiv \mathbf{W}\mathbb{C}\mathbf{P}^1_{2,N-2} , \tag{2.18b}$$
$$\mathcal{M}^2_{r<0} := \Big\{ (p_2; s_i) \in \mathbb{C}^{N+1} \Big| \mathcal{D} = G_2 = 0 , \ r < 0 \Big\} \Big/ U(1) . \tag{2.18c}$$

These two subspaces are glued in the region given by $p_1 = {}^\forall s_i = 0$. Although this situation is same as to the case of $3 \le \ell \le N-1$, the appearing massless effective theories are quite different.

Here let us analyze the effective theories on the subspace $\mathcal{M}^1_{r<0} = \mathbf{W}\mathbb{C}\mathbf{P}^1_{2,N-2}$. We choose a point in this subspace as a supersymmetric vacuum and take VEVs of all scalar fields. Then we make all scalar fields fluctuate around the VEVs. Fluctuation modes $\widetilde{p}_1$ and $\widetilde{p}_2$ are subject to the constraints

such that they move only tangent to $\mathbf{W}\mathbb{C}\mathbf{P}^1_{2,N-2}$. The fluctuation modes $\widetilde{s}_i$ have no degrees of freedom because of the variation of the constraint $p_1 \partial_i G_2 = 0$. (In the case of (2.13b), the equations $G_\ell = 0$ and $p_1 \partial_i G_\ell = 0$ are trivially satisfied in $\mathbf{W}\mathbb{C}\mathbf{P}^1_{\ell,N-\ell}$. These variations are also trivial. However the case of $\ell = 2$ is quite different. By definition, some $\partial_i \partial_j G_2$ must have non-zero values. Thus even though the above equations are trivially satisfied in the subspace $\mathbf{W}\mathbb{C}\mathbf{P}^1_{2,N-2}$, their variations give non-trivial constraints on the fluctuation modes.) Under these conditions we write down the potential energy density (2.8) in terms of VEVs $\langle \varphi_a \rangle$ and fluctuation modes $\widetilde{\varphi}_a$ and $\widehat{\varphi}_a$:

$$
\begin{aligned}
\mathcal{U} \ &= \ \frac{e^2}{2} \Big\{ 2 \, \mathrm{Re} \Big[ 2 \widehat{p}_1 \langle \overline{p}_1 \rangle + (N-2) \widehat{p}_2 \langle \overline{p}_2 \rangle \Big] - \sum_i \big| \widehat{s}_i \big|^2 + 2 \big| \widetilde{p}_1 + \widehat{p}_1 \big|^2 + (N-2) \big| \widetilde{p}_2 + \widehat{p}_2 \big|^2 \Big\}^2 \\
&\qquad + \big| G_2(\widehat{s}) \big|^2 + \big| \langle p_1 \rangle + \widetilde{p}_1 + \widehat{p}_1 \big|^2 \cdot \sum_i \big| \partial_i G_2(\widehat{s}) \big|^2 \\
&\qquad + 2 |\widehat{\sigma}|^2 \Big\{ \sum_i \big| \widehat{s}_i \big|^2 + 4 \big| \langle p_1 \rangle + \widetilde{p}_1 + \widehat{p}_1 \big|^2 + (N-2)^2 \big| \langle p_2 \rangle + \widetilde{p}_2 + \widehat{p}_2 \big|^2 \Big\} \, .
\end{aligned}
$$

This function denotes the following: The fluctuation modes $\widehat{s}_i$, $\widehat{p}_1$ and $\widehat{p}_2$ acquire masses $m^2 = \mathcal{O}(e^2 |r|)$; the modes $\widetilde{p}_1$ and $\widetilde{p}_2$ remain massless and move tangent to $\mathbf{W}\mathbb{C}\mathbf{P}^1_{2,N-2}$. Thus taking $e \to \infty$ and $|r| \to \infty$, we obtain the massless effective theory described by

$$
\mathcal{N} = (2,2) \text{ supersymmetric NLSM on } \mathbf{W}\mathbb{C}\mathbf{P}^1_{2,N-2} \, . \tag{2.19}
$$

This sigma model has $\mathbb{Z}_\alpha$ orbifold symmetry coming from the property of $\mathbf{W}\mathbb{C}\mathbf{P}^1_{2,N-2}$, where $\alpha = \mathrm{GCM}\{2, N-2\}$. This effective theory does not include massless LG theory. The reason is that the degree two polynomial $G_2$ generates mass terms such as $|\langle p_1 \rangle|^2 \sum_i |\partial_i G_2|^2$. (See, for example, [31].)

Now we consider the effective theory on two specific points in $\mathbf{W}\mathbb{C}\mathbf{P}^1_{2,N-2}$ like (2.15) and (2.16). Expanding the theory on the one point $(p_1, p_2) = (p_1, 0)$, the gauge symmetry is partially restored to $\mathbb{Z}_2$. Thus we obtain the effective theory on this specific point as

$$
\mathcal{N} = (2,2) \text{ SCFT on } \mathbb{C}^1 / \mathbb{Z}_2 \, . \tag{2.20}
$$

Note that this theory can possess the LG theory with a quadratic superpotential $W_{\mathrm{LG}} = \langle p_1 \rangle G_2(S)$, which gives massive modes of $S_i$.

The effective theory drastically changes if we expand the theory on another point $(p_1, p_2) = (0, p_2)$ in $\mathbf{W}\mathbb{C}\mathbf{P}^1_{2,N-2}$. On this point, the broken gauge symmetry is enhanced to $\mathbb{Z}_{N-2}$ and the fluctuation modes $\widehat{s}_i$ become massless with being free from any constraints. Both $\widetilde{p}_1$ and $\widehat{p}_1$ are massless and linearly combined in the potential energy density. The remaining field $\widetilde{p}_2$ becomes zero because there exists a non-trivial variation of the constraint $\mathcal{D} = 0$. Summarizing these results, we find that the following massless effective theory appears in the limit $e, |r| \to \infty$:

$$
\Big\{ \mathcal{N} = (2,2) \ \text{"LG" theory with } W_{\mathrm{LG}} = P_1 \cdot G_2(S) \ \text{ on } \mathbb{C}^{N+1} \Big\} \Big/ \mathbb{Z}_{N-2} \, . \tag{2.21}
$$

Although this superpotential also has no isolated singularities, this theory should describe a non-trivial SCFT. We shall return here in later discussions.

We next study the massless effective theories on the subspace $\mathcal{M}^2_{r<0}$ defined in (2.18c). The potential energy density is obtained as

$$\begin{aligned}
\mathcal{U} \ &= \ \frac{e^2}{2}\Big\{ 2\,\mathrm{Re}\Big[ -\sum_i \widehat{s}_i \langle \overline{s}_i \rangle + (N-2)\widehat{p}_2 \langle \overline{p}_2 \rangle \Big] - \sum_i \big| \widetilde{s}_i + \widehat{s}_i \big|^2 + 2\big|\widehat{p}_1\big|^2 + (N-2)\big|\widetilde{p}_2 + \widehat{p}_2\big|^2 \Big\}^2 \\
&\quad + \Big| \sum_i \widehat{s}_i \, \partial_i G_2(\langle s \rangle) + \frac{1}{2!} \sum_{i,j} (\widetilde{s} + \widehat{s})_i (\widetilde{s} + \widehat{s})_j \cdot \partial_i \partial_j G_2(\langle s \rangle) \Big|^2 \\
&\quad + |\widehat{p}_1|^2 \cdot \sum_i \Big| \partial_i G_2(\langle s \rangle) + \sum_j (\widetilde{s} + \widehat{s})_j \cdot \partial_i \partial_j G_2(\langle s \rangle) \Big|^2 \\
&\quad + 2|\widehat{\sigma}|^2 \Big\{ \sum_i \big| \langle s_i \rangle + \widetilde{s}_i + \widehat{s}_i \big|^2 + 4\big|\widehat{p}_1\big|^2 + (N-2)^2 \big| \langle p_2 \rangle + \widetilde{p}_2 + \widehat{p}_2 \big|^2 \Big\}
\end{aligned}$$

under the following constraints on fluctuation modes: The fluctuations $\widetilde{s}_i$ and $\widetilde{p}_2$ move tangent to $\mathcal{M}^2_{r<0}$; the other tangent mode $\widetilde{p}_1$ is zero; the fluctuations $\widehat{\varphi}_a$ are all massive of $m^2 = \mathcal{O}(e^2|r|)$. Thus the effective theory expanded around generic points in $\mathcal{M}^2_{r<0}$ becomes

$$\mathcal{N} = (2,2) \text{ supersymmetric NLSM on } \mathcal{M}^2_{r<0} \tag{2.22}$$

in the IR and large volume limit: $e, |r| \to \infty$. The $U(1)$ gauge symmetry is completely broken if some $\langle s_i \rangle \neq 0$ exist. On the other hand, if we expand the theory on a specific point $p_1 = {}^\forall s_i = 0$, the gauge symmetry is partially restored to $\mathbb{Z}_{N-2}$.

So far we have studied the effective theories on all regions of the vacuum manifold $\mathcal{M}_{\text{orbifold}}$ of $\ell = 2$. From the same reason discussed in the case of $3 \leq \ell \leq N-1$, there exists a phase transition between the theories (2.21) and (2.22) because of the non-trivial constraint coming from the variation of the equation $G_2 = 0$. Thus we find that the GLSM for the $\mathcal{O}(-N+2)$ bundle on $\mathbb{C}\mathbf{P}^{N-1}[2]$ also includes two phases in the negative FI parameter region. The phase on (2.19) is called the orbifold phase, and we refer the phase on (2.22) to the 3rd phase.

Here we illustrate the relation among the phases in the GLSM schematically in Figure 1:

In the large volume limit $|r| \to \infty$, these three effective theories (2.11), (2.14) and (2.17) become well-defined. In later discussions we shall consider how these effective theories deform in the small FI parameter limit $|r| \to 0$. There we must consider the singular phase [14].

## Effective theory of $\ell = 1$

Finally we investigate the $\ell = 1$ case. Since the polynomial $G_{\ell=1}(S)$ is of degree one, there exist some non-zero values of $\partial_i G_1(S)$. Thus, combining this condition with the other constraints which define

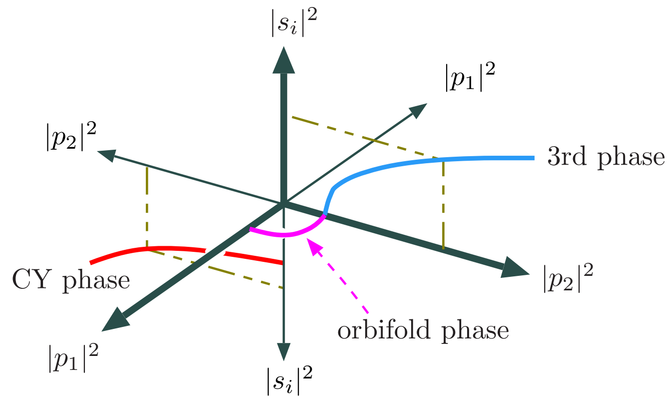


Figure 1: *Various phases in GLSM for $\mathcal{O}(-N+\ell)$ bundle on $\mathbb{C}\mathbf{P}^{N-1}[\ell]$ with $2 \le \ell \le N-1$. The axes with thin/thick lines represent the vacuum space coordinates in the positive/negative FI parameter regions, respectively.*

$\mathcal{M}_{\text{orbifold}}$, we find that $p_1$ must be zero and obtain the following reduced vacuum manifold:

$$\mathcal{M}_{\text{orbifold}}\big|_{\ell=1} \ = \ \Big\{ (p_2; s_i) \in \mathbb{C}^{N+1} \, \Big| \, \mathcal{D} = G_1 = 0 \, , \ r < 0 \Big\} \Big/ U(1) \ =: \ \mathcal{M}^2_{r<0} \, . \tag{2.23}$$

Since this space is defined in the same way as (2.13c) and (2.18c), we also referred it to $\mathcal{M}^2_{r<0}$.

After taking VEVs of scalar fields which live in (2.23), we make scalar fields fluctuate around the VEVs. These fluctuation modes are subject to constraints: $\widetilde{s}_i$ and $\widetilde{p}_2$ move only tangent to $\mathcal{M}^2_{r<0}$; $\widetilde{p}_1$ is zero. Substituting these into (2.8), we obtain the expanded potential energy density

$$\begin{aligned}
\mathcal{U} \ &= \ \frac{e^2}{2} \Big\{ 2\,\mathrm{Re}\Big[ - \sum_i \widehat{s}_i \langle \overline{s}_i \rangle + (N-1) \widehat{p}_2 \langle \overline{p}_2 \rangle \Big] - \sum_i \big| \widetilde{s}_i + \widehat{s}_i \big|^2 + \big| \widehat{p}_1 \big|^2 + (N-1) \big| \widetilde{p}_2 + \widehat{p}_2 \big|^2 \Big\}^2 \\
&\quad + \big| G_1(\widehat{s}) \big|^2 + \big| \widehat{p}_1 \big|^2 \cdot \sum_i \big| \partial_i G_1(\langle s \rangle) \big|^2 \\
&\quad + 2 |\widehat{\sigma}|^2 \Big\{ \sum_i \big| \langle s_i \rangle + \widetilde{s}_i + \widehat{s}_i \big|^2 + \big| \widehat{p}_1 \big|^2 + (N-1) \big| \langle p_2 \rangle + \widetilde{p}_2 + \widetilde{p}_2 \big|^2 \Big\} \, .
\end{aligned}$$

This indicates that the modes $\widetilde{s}_i$ and $\widetilde{p}_2$ remain massless whereas the modes $\widehat{s}_i$, $\widehat{p}_1$ and $\widehat{p}_2$ become massive. Thus the following massless effective theory appears in the limit $e, |r| \to \infty$:

$$\mathcal{N} = (2,2) \text{ supersymmetric NLSM on } \mathcal{M}^2_{r<0} \, , \tag{2.24}$$

where the $U(1)$ gauge symmetry is completely broken because ${}^{\exists}\langle s_i \rangle \neq 0$. While if we set the VEVs to ${}^{\forall}\langle s_i \rangle = 0$, the broken $U(1)$ gauge symmetry is enhanced to $\mathbb{Z}_{N-1}$. In addition the modes $\widehat{s}_i$ become massless and are combined with the tangent modes $\widetilde{s}_i$, which are still under constraint $G_1 = 0$. The mode $\widetilde{p}_2$ becomes zero, which is derived from the variation of $\mathcal{D} = 0$. In this specific point we can see

that the space $\mathcal{M}^2_{r<0}$ is deformed to $\mathbb{C}^{N-1}/\mathbb{Z}_{N-1}$ and the effective theory are

$$\mathcal{N} = (2,2) \text{ SCFT on } \mathbb{C}^{N-1}/\mathbb{Z}_{N-1} \, . \tag{2.25}$$

The two effective theories (2.24) and (2.25) are smoothly connected without any phase transitions coming from the variations of constraints. Thus we find that in the $\ell = 1$ case there exists only one phase in the negative FI parameter region. We refer this to the orbifold phase, as usual. Here we illustrate the schematic relation between the CY phase and the orbifold phase in Figure 2:

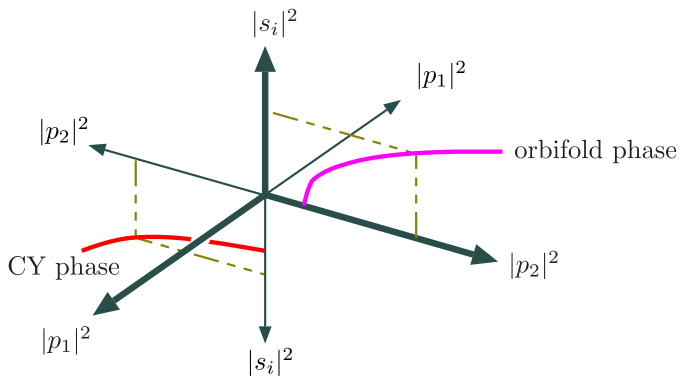


Figure 2: *Various phases in GLSM for $\mathcal{O}(-N+1)$ bundle on $\mathbb{C}\mathbf{P}^{N-1}[1]$.*

Note that the GLSM for $\mathcal{O}(-N+1)$ bundle on $\mathbb{C}\mathbf{P}^{N-1}[1]$ is completely the same as the one for $\mathcal{O}(-N+1)$ bundle on $\mathbb{C}\mathbf{P}^{N-2}$. This is because the hypersurface $\mathbb{C}\mathbf{P}^{N-1}[1]$ is nothing but $\mathbb{C}\mathbf{P}^{N-2}$. Thus the vacuum structure and phases are also equal to each other.

### 2.5 Singularity phase

In this subsection let us analyze the singularity phase. As mentioned before, the effective theory (2.11) becomes singular if $r \to +0$. The effective theories in the orbifold and the 3rd phases also become singular if $r \to -0$. Thus we will study the singularity phase $r = 0$ in order to avoid the singularities in effective theories. In this analysis we will find that there appears a new branch. Then we will discuss how to avoid the singularity.

Here we study how the vacuum manifold (2.9) is reduced in the $r = 0$ phase. If we assume $p_1 \neq 0$, then we obtain $\sum_i |s_i|^2 \neq 0$ from $\mathcal{D} = 0$. However the equations $G_\ell(s) = 0$ and $p_1 \partial_i G_\ell(s) = 0$ insist that all $s_i$ vanish. This is a contradiction. Thus $p_1$ must be zero. Under this condition, we obtain two solutions. One is obtained by $\mathcal{D} = 0$ and $\sigma = 0$. In general this solution has non-zero $\phi_a$, where $\phi_a$ are scalar component fields of chiral superfields. The other solution is given by ${}^\forall\phi_a = 0$ and $\sigma$ is

free. We refer the former and latter solutions to the Higgs and Coulomb branches, respectively. These branches are similar to the ones of $\mathcal{N}=2$ SQCD in four dimensions [28]. (The CY, orbifold and 3rd phases are all in the Higgs branch.) They are connected if all scalar component fields vanish: ${}^{\forall}\varphi_a = 0$. Now we analyze the effective theories on these two branches.

### Higgs and Coulomb branches

Let us consider the Higgs and Coulomb branches in detail. In the Higgs branch, there exist two supersymmetric vacuum solutions. One is

$$ {}^{\forall}\langle\phi_a\rangle \ = \ \mathcal{O}(|r|) \ \to \ 0 \, , \qquad \langle\sigma\rangle \ = \ 0 \, . \tag{2.26a} $$

This solution is smoothly connected with the supersymmetric vacuum solutions in the phases of non-vanishing FI parameter. The other is

$$ \langle p_1\rangle \ = \ 0 \, , \qquad {}^{\forall}\langle s_i\rangle \, , \langle p_2\rangle \ : \quad \text{arbitrary order} \, , \qquad \langle\sigma\rangle \ = \ 0 \, , \tag{2.26b} $$

which is satisfied only on $r=0$. Although the first solution (2.26a) appears in each GLSM, the second solution (2.26b) does not satisfy the supersymmetric vacuum condition $\mathcal{U}(\varphi)=0$ in some GLSMs, for example, the GLSM for $\mathbb{C}\mathbf{P}^{N-1}[N]$.

In the Coulomb branch, we can set that ${}^{\forall}\langle\phi_a\rangle = 0$ and $\langle\sigma\rangle$ is free. This solution appears only when the FI parameter vanishes. If we choose $\langle\sigma\rangle$ to be zero, i.e., all the VEVs of scalar fields vanish $\langle\varphi_a\rangle = 0$, the Coulomb branch connects with the Higgs branch.

Let us consider a massless effective theory in the Coulomb branch. Since the scalar field $\sigma$ has mass dimension one, we take the VEV $\langle\sigma\rangle$ to be very large. Owing to this, all chiral superfields $S_i$, $P_1$ and $P_2$ acquire very large masses via $\mathcal{U}_\sigma(\varphi)$ in (2.8). Taking $\langle\sigma\rangle\to\infty$ and integrating out all massive fields[1], we obtain the following effective Lagrangian:

$$ \begin{aligned} \mathscr{L}_{\text{eff}} \ &= \ \int \mathrm{d}^4\theta \, \big\{ -\mathcal{K}_{\text{eff}}(\Sigma,\overline{\Sigma})\big\} + \Big( \frac{1}{\sqrt{2}} \int \mathrm{d}^2\widetilde{\theta} \, \widetilde{W}_{\text{eff}}(\Sigma) + c.c. \Big) \, , \\ \widetilde{W}_{\text{eff}}(\Sigma) \ &= \ -\Sigma \, t - \sum_a Q_a\Sigma \Big\{ \log\Big(\frac{Q_a\Sigma}{\mu}\Big) - 1 \Big\} \\ &= \ -\Sigma\Big( t - \ell\log(-\ell) - (N-\ell)\log(-N+\ell)\Big) \, . \end{aligned} $$

Note that the twisted superpotential was deformed by the quantum effects coming from the integration of massive fields. This effective Lagrangian presents the asymptotic form of the potential energy density

$$ \mathcal{U}_{\text{eff}}(\sigma) \ = \ \frac{e_{\text{eff}}^2}{2}\big|\partial_\sigma\widetilde{W}_{\text{eff}}(\sigma)\big|^2 \ = \ \frac{e_{\text{eff}}^2}{2}\Big| t - \ell\log(-\ell) - (N-\ell)\log(-N+\ell)\Big|^2 \, . \tag{2.27} $$

[1]Here we can integrate out $\Phi_a$ because the superpotential $W_{\text{GLSM}}$ does not contribute to the deformation of the effective twisted superpotential $\widetilde{W}_{\text{eff}}(\Sigma)$. For the precise derivation, see chapter 15 in [17].

In order that the effective theory remains supersymmetric, the potential energy density must be zero in a specific value of $\sigma$. Notice that if the complexified FI parameter is given by

$$t \;=\; \ell \log(-\ell) + (N-\ell) \log(-N+\ell) \ , \tag{2.28}$$

the potential energy density becomes always zero. If it happens, the effective theory does not have any mass gap and becomes singular as a two-dimensional field theory. Thus (2.28) is the quantum singular point of the GLSM. In the classical point of view, the value $t = 0$ looks like a singular point in the theory. Integrating out the massive fields, we find that the singular point moves to (2.28). The massless effective theories in Coulomb and Higgs branches are connected with each other avoiding this singular point.

### CY/LG correspondence and topology change

As mentioned before, the massless effective theories are only valid if we take the FI parameter to be infinitely large $|r| \to \infty$. In this limit the effective theories are (partly) described by the NLSMs. However if we change the FI parameter to be small, the NLSM representations are no longer well-defined and must be deformed. This phenomenon has been already studied in [14] as following: If the FI parameter goes to zero $r \to 0$, the effective theory on the CY phase moves to the theory on the Coulomb branch in the singularity phase avoiding the singular point. Furthermore the effective theory connects to the LG theory in the orbifold phase when $r \to -\infty$.

The above phenomenon suggests that, rather than the LG theory being equivalent to the sigma model on the CY manifold, they are two different phases of the same system, i.e., the system of the single GLSM. Thus the CY/LG correspondence [32, 33, 34, 35, 36, 37, 38] can be read from the phase transition. In fact, it has been proved that the sigma model on $\mathbb{C}\mathbf{P}^4[5]$ and the $\mathbb{Z}_5$-orbifolded LG theory with $W_{\rm LG} = G_5(S)$, which are equivalent to each other, appear as the distinct phases in the single GLSM. Furthermore the topology change is also understood in the framework of the phase transition of the GLSM. The flop of the resolved conifold $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{C}\mathbf{P}^1$ is a typical example [14].

Now let us apply the above discussions to the GLSM for $\mathcal{O}(-N+\ell)$ bundle on $\mathbb{C}\mathbf{P}^{N-1}[\ell]$. For example we consider the relations among the various phases of $3 \le \ell \le N-1$, where we have found three phases: The CY phase on $\mathcal{M}_{\rm CY}$, the orbifold phase on $\mathcal{M}^1_{r<0}$ and the 3rd phase on $\mathcal{M}^2_{r<0}$. Furthermore we have found four effective theories. Let us discuss the relations among them:

- The effective theory on the CY phase (2.11) and the theory on the 3rd phase (2.17) are related to each other via a topology change, because the defining equations of the target spaces are equal

except for the sign of the FI parameter. Furthermore these two effective theories are both sigma models, which do not include the potential theory sectors such as a LG theory.

- (2.16) and (2.17) are connected at the point $p_1 = {}^{\forall}s_i = 0$ in $\mathcal{M}_{\text{orbifold}}$. Since the former is the sigma model and the latter is the LG theory, there exists a phase transition between these two theories, which are equivalent to each other by the CY/LG correspondence.
- Both the theories (2.15) and (2.16) are on the weighted projective space and are included in the theory (2.14).
- The LG theory (2.15) is equivalent to the sigma model (2.11) by the CY/LG correspondence.

Notice that the sigma model (2.11) is not related to (2.16) directly, because the theory (2.16) has already been connected to (2.17) while the CY/LG correspondence connects between two theories by one-to-one. These connections are realized through the singularity phase. Even though we wrote down the connections only from the qualitative point of view, we can acquire non-trivial relations among the effective theories as illustrated in Figure 3:

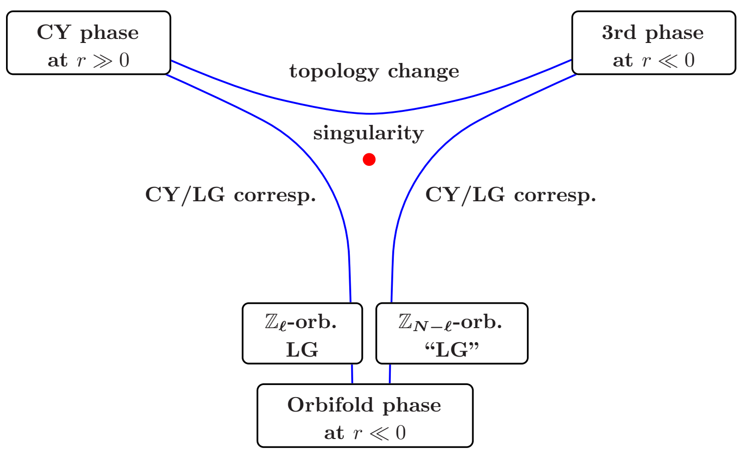


Figure 3: *The relation among various phases around the singularity: a conjecture.*

In the case of compact CY manifolds, we have already understood that the local rings in the LG theory are identified with the chiral rings of the SCFT and these chiral rings are related to the harmonic forms on such manifolds [15]. However we have no proof that this relation is also satisfied in the case of noncompact CY manifolds. Thus we must investigate the spectra of the above effective theories as a future problem.

As discussed before, we have obtained various massless effective theories by decomposing all massive modes. Thus they are just approximate descriptions which must be deformed if we can exactly

integrate out massive modes. In the next section we will study the T-dual theory of the GLSM [16]. This formulation is so powerful to obtain the exact effective theories. Analyzing them exact theories we will re-investigate the massless effective theories in the original GLSM.

# 3 T-dual theory

In this section we consider T-dual of GLSMs. It is quite significant to study it because we can obtain exact descriptions of the low energy effective theories. Furthermore they will also indicate how the exact effective theories are realized in the original GLSM. In fact, in the original GLSM, we obtained just approximate effective theories. There we did not perform integrating-out but just decomposed all massive modes because it is generally impossible to integrate them out. In the model proposed in [16], we calculate a function which is directly related to the partition function. Thus we will obtain exact effective theories as quantum field theory.

## 3.1 General construction

Here we briefly review the T-duality of a generic GLSM without any superpotentials [16]. We start from the following Lagrangian in two-dimensional worldsheet:

$$\mathscr{L}' = \int d^4\theta \Big\{ -\frac{1}{e^2}\overline{\Sigma}\Sigma + \sum_a \Big( e^{2Q_a V + B_a} - \frac{1}{2}(Y_a + \overline{Y}_a) B_a \Big) \Big\} + \Big( \frac{1}{\sqrt{2}} \int d^2\widetilde{\theta}\, (-\Sigma\, t) + (c.c.) \Big) , \tag{3.1}$$

where $Y_a$ are twisted chiral superfields whose imaginary parts are periodic of period $2\pi$. We incorporate real superfields $B_a$ as auxiliary fields.

Integrating out twisted chiral superfields $Y_a$, we obtain $\overline{D}_+ D_- B_a = D_+ \overline{D}_- B_a = 0$, whose solutions are written in terms of chiral superfields $\Psi_a$ and $\overline{\Psi}_a$ such as $B_a = \Psi_a + \overline{\Psi}_a$. When we substitute them into the Lagrangian (3.1), a GLSM Lagrangian appears:

$$\begin{aligned} \mathscr{L}'\Big|_{B_a = \Psi_a + \overline{\Psi}_a} &= \int d^4\theta \Big\{ -\frac{1}{e^2}\overline{\Sigma}\Sigma + \sum_a \overline{\Phi}_a\, e^{2Q_a V} \Phi_a \Big\} + \Big( \frac{1}{\sqrt{2}} \int d^2\widetilde{\theta}\, (-\Sigma\, t) + (c.c.) \Big) \\ &\equiv \mathscr{L}_{\rm GLSM} , \end{aligned} \tag{3.2}$$

where we re-wrote $\Phi_a := e^{\Psi_a}$. On the other hand, when we first integrate out $B_a$ in the original Lagrangian (3.1), we obtain

$$B_a = -2Q_a V + \log \Big( \frac{Y_a + \overline{Y}_a}{2} \Big) . \tag{3.3}$$

Let us insert these solutions into (3.1). By using a deformation

$$\int \mathrm{d}^4\theta\, Q_a V Y_a \;=\; -\frac{1}{2} Q_a \int \mathrm{d}^2\widetilde{\theta}\, \overline{D}_+ D_- V Y_a \;=\; -\frac{1}{\sqrt{2}} Q_a \int \mathrm{d}^2\widetilde{\theta}\, \Sigma Y_a \ ,$$

we find that a Lagrangian of twisted chiral superfields appears:

$$\mathscr{L}_{\mathrm{T}} \;=\; \int d^4\theta \Big\{ -\frac{1}{e^2}\overline{\Sigma}\Sigma - \sum_a \Big( \frac{1}{2}(Y_a + \overline{Y}_a) \log(Y_a + \overline{Y}_a) \Big) \Big\} + \Big( \frac{1}{\sqrt{2}} \int \mathrm{d}^2\widetilde{\theta}\, \widetilde{W} + (c.c.) \Big) \ , \tag{3.4a}$$

$$\widetilde{W} \;=\; \Sigma \Big( \sum_a Q_a Y_a - t \Big) + \mu \sum_a \mathrm{e}^{-Y_a} \ . \tag{3.4b}$$

This Lagrangian is T-dual of the gauge theory (3.2). Notice that the twisted superpotential $\widetilde{W}$ is corrected by instanton effects where the instantons are the vortices of the gauge theory. In attempt to analyze a model satisfying $\sum_a Q_a = 0$, the scale parameter $\mu$ is omitted by field re-definitions. Relations between chiral superfields $\Phi_a$ in (3.2) and twisted chiral superfields $Y_a$ in (3.4) are

$$2\overline{\Phi}_a\, \mathrm{e}^{2Q_a V} \Phi_a \;=\; Y_a + \overline{Y}_a \ . \tag{3.5}$$

We can see that the shift symmetry $Y_a \equiv Y_a + 2\pi i$ comes from the $U(1)$ rotation symmetry on $\Phi_a$. In the IR limit $e \to \infty$, $\Sigma$ becomes non-dynamical and generates a following constraint from $\widetilde{W}$:

$$\sum_a Q_a Y_a \;=\; t \ ,$$

which corresponds to the condition $\mathcal{D} = 0$ in the original GLSM.

In this formulation it is convenient to incorporate a function defined by

$$\Pi \;:=\; \int \mathrm{d}\Sigma \prod_a \mathrm{d}Y_a\, \exp\big( -\widetilde{W} \big) \ , \tag{3.6}$$

where $\widetilde{W}$ is defined in (3.4b). When we consider low energy effective theories of the theory (3.4), we take the gauge coupling constant to be infinity $e \to \infty$. In this limit $\Sigma$ is no longer dynamical and becomes just an auxiliary field. Thus the function (3.6) is re-written by integrating-out of $\Sigma$:

$$\Pi \;=\; \int \prod_a \mathrm{d}Y_a\, \delta\Big( \sum_a Q_a Y_a - t \Big)\, \exp\Big( -\sum_a \mathrm{e}^{-Y_a} \Big) \ .$$

Via suitable field re-definitions, we can read a LG theory of twisted chiral superfields. Moreover we also obtain a period integral of a "mirror pair" of the manifold which appeared in the effective theories in the original GLSM. Thus we often refer the function (3.6) to the period integral.

Suppose the theory is topologically $A$-twisted [39]. In the topologically $A$-twisted theory, twisted chiral superfields are only valid while the other fields such as chiral superfields and real superfields are all BRST-exact. Due to this the Lagrangian is reduced only to the twisted superpotential and the

partition function is obtained as the integral of weight $\mathrm{e}^{-\widetilde{W}}$. This is nothing but the period integral defined in (3.6). Thus as far as considering the $A$-twisted sector, the effective theories derived from this function are exact.

Unfortunately, no one knows an exact formulation of a T-dual Lagrangian $\mathscr{L}_{\mathrm{T}}$ of a GLSM with a generic superpotential $W_{\mathrm{GLSM}}$. This is partly because the above formulation is only powerful when we consider T-duality of topologically $A$-twisted GLSMs. As mentioned above, any deformations of $W_{\mathrm{GLSM}}$ are BRST-exact in the topological $A$-twisted theory. However, even though in the $A$-theories, we can analyze T-dual theories of specific GLSMs with superpotentials of type $W_{\mathrm{GLSM}} = P \cdot G_{\ell}(S)$, where $G_{\ell}(S)$ is a homogeneous polynomial of degree $\ell$ with respect to chiral superfields $S$. In order to do this, let us deform the above period integral (3.6) to

$$\widehat{\Pi} \ = \ \int \mathrm{d}\Sigma \prod_a \mathrm{d}Y_a \, (\ell \Sigma) \, \exp \big( - \widetilde{W} \big) \, . \tag{3.7}$$

This function can be derived through the discussions of Cecotti and Vafa [40], and Morrison and Ronen Plesser [41]. Here we omit a precise derivation. Please see it in [16]. In this formulation we also take the IR limit $e \to \infty$ and integrate out the superfield $\Sigma$, because we want to obtain the mirror dual descriptions of the effective theories of the original GLSM with superpotential. However the factor $\ell\Sigma$ in (3.7) prevents from exact integrating-out of $\Sigma$. Thus we need to replace this factor to other variable which does not disturb the integration. If we wish to obtain the LG description we replace the factor to the differential with respect to the FI parameter such as $\ell\Sigma \to \ell \frac{\partial}{\partial t}$. On the other hand when we derive the mirror geometry we replace this to the differential operator of an appropriate twisted chiral superfield derived from the negatively charged chiral superfield, for example, $\ell\Sigma \to \frac{\partial}{\partial Y_P}$, where $Y_P$ is the twisted chiral superfield of the chiral superfield $P$ of charge $-\ell$. The resulting geometry has a $\mathbb{Z}_{\ell}$-type orbifold symmetry. For example, we start from the GLSM for quintic hypersurface $\mathbb{C}\mathbf{P}^4[5]$. Performing T-duality and taking the IR limit, we obtain not only the mirror dual geometry $\mathbb{C}\mathbf{P}^4[5]/(\mathbb{Z}_5)^3$ but also its LG description [16]. This procedure is so powerful that we develop it in order to obtain the mirror descriptions of the noncompact CY manifolds. In subsections 3.3 and 3.4 we will study how to obtain LG theories defined by the twisted superpotential and mirror dual geometries, respectively. We can also obtain another geometry with a different orbifold symmetry if we replace $\ell\Sigma$ to the differential of other suitable twisted chiral superfield.

### 3.2 Field configuration

Let us analyze the T-dual theory of the GLSM for the $\mathcal{O}(-N+\ell)$ bundles on $\mathbb{C}\mathbf{P}^{N-1}[\ell]$. The field configuration is assigned as follows:

$$
\begin{array}{c|ccccc}
\hline
\text{chiral superfield } \Phi_a & S_1 & \ldots & S_N & P_1 & P_2 \\
\hdashline
U(1) \text{ charge } Q_a & 1 & \ldots & 1 & -\ell & -N+\ell \\
\hdashline
\text{twisted chiral superfield } Y_a & Y_1 & \ldots & Y_N & Y_{P_1} & Y_{P_2} \\
\hline
\end{array}
\tag{3.8}
$$

The twisted chiral superfields $Y_a$ are periodic variables $Y_a \equiv Y_a + 2\pi i$. They are defined from the chiral superfields $\Phi_a$ via (3.5). As we have already discussed, the twisted superpotential $\widetilde{W}$ and the period integral $\widehat{\Pi}$, given by the followings, play key roles:

$$
\widetilde{W} \ = \ \Sigma\Big(\sum_{i=1}^N Y_i - \ell Y_{P_1} - (N-\ell)Y_{P_2} - t\Big) + \sum_{i=1}^N \mathrm{e}^{-Y_i} + \mathrm{e}^{-Y_{P_1}} + \mathrm{e}^{-Y_{P_2}} \ , \tag{3.9a}
$$

$$
\widehat{\Pi} \ = \ \int \mathrm{d}\Sigma \prod_{i=1}^N \mathrm{d}Y_i \, \mathrm{d}Y_{P_1} \, \mathrm{d}Y_{P_2} \, (\ell\Sigma) \, \exp\big(-\widetilde{W}\big) \ . \tag{3.9b}
$$

Let us take the IR limit $e \to \infty$ in order to consider the low energy effective theories. It is clear that the dynamics of $\Sigma$ is frozen and this superfield becomes just an auxiliary superfield. Thus we must replace the factor $\ell\Sigma$ in the period integral (3.9b) to appropriate variables.

### 3.3 Mirror Landau-Ginzburg descriptions

In this subsection we will derive LG theories with orbifold symmetries. In order to do this, we change the variable $\ell\Sigma$ in the period integral (3.9b) to

$$
\ell\Sigma \ \to \ \ell\frac{\partial}{\partial t} \ .
$$

This replacing can be easily performed because of the existence of the term $\Sigma(\sum_a Q_a Y_a - t)$ in $\widehat{\Pi}$. Then we integrate out the superfield $\Sigma$ and obtain

$$
\widehat{\Pi} \ = \ \ell\frac{\partial}{\partial t} \int \prod_{i=1}^N \mathrm{d}Y_i \, \mathrm{d}Y_{P_1} \, \mathrm{d}Y_{P_2} \, \delta\Big(\sum_i Y_i - \ell Y_{P_1} - (N-\ell)Y_{P_2} - t\Big) \exp\Big(-\sum_i \mathrm{e}^{-Y_i} - \mathrm{e}^{-Y_{P_1}} - \mathrm{e}^{-Y_{P_2}}\Big) \ . \tag{3.10}
$$

Next let us solve the $\delta$-function in this function. We note that there are two ways to solve it. One is to write the variable $Y_{P_1}$ in terms of $Y_i$ and $Y_{P_2}$. The other is to solve $Y_{P_2}$ by $Y_i$ and $Y_{P_1}$. Both two solutions give consistent LG theories with orbifold symmetries.

## Solution one: $\mathbb{Z}_\ell$ orbifolded LG theory

Let us solve the variable $Y_{P_1}$ via the $\delta$-function in (3.10):

$$-Y_{P_1} \;=\; \frac{1}{\ell}\Big(t - \sum_{i=1}^{N} Y_i + (N-\ell) Y_{P_2}\Big)\;.$$

Performing the $t$-derivative in (3.10) after the substitution of this solution, we obtain

$$\widehat{\Pi} \;=\; \mathrm{e}^{t/\ell} \int \prod_{i=1}^{N} \big(\mathrm{e}^{-\frac{1}{\ell}Y_i}\mathrm{d}Y_i\big)\,\big(\mathrm{e}^{\frac{N-\ell}{\ell}Y_{P_2}}\mathrm{d}Y_{P_2}\big)\;\exp\Big(-\sum_i \mathrm{e}^{-Y_i} - \mathrm{e}^{t/\ell}\prod_i \mathrm{e}^{-\frac{1}{\ell}Y_i}\mathrm{e}^{\frac{N-\ell}{\ell}Y_{P_2}} - \mathrm{e}^{-Y_{P_2}}\Big)\;.$$

It is clear that the integral measure is not canonical. Transforming the variables into $X_i^{\ell} := \mathrm{e}^{-Y_i}$ and $X_{P_2}^{\ell} := \mathrm{e}^{(N-\ell)Y_{P_2}}$, we obtain the following period integral with a canonical measure up to an overall constant[2]:

$$\widehat{\Pi} \;=\; \int \prod_{i=1}^{N} \mathrm{d}X_i\,\mathrm{d}X_{P_2}\;\exp\Big(-\sum_i X_i^{\ell} - X_{P_2}^{-\frac{\ell}{N-\ell}} - \mathrm{e}^{t/\ell} X_1 \cdots X_N X_{P_2}\Big)\;. \tag{3.11}$$

Since $Y_a$ are periodic with respect to the shifts of their imaginary parts $Y_a \equiv Y_a + 2\pi i$, the new variables $X_i$ and $X_{P_2}$ are symmetric under the following phase shifts:

$$X_i \;\mapsto\; \omega_i X_i\;, \qquad X_{P_2} \;\mapsto\; \omega_{P_2} X_{P_2}\;, \qquad \omega_i^{\ell} \;=\; \omega_{P_2}^{-\frac{\ell}{N-\ell}} \;=\; \omega_1\omega_2\cdots\omega_N\omega_{P_2} \;=\; 1\;. \tag{3.12}$$

We can read from (3.11) and (3.12) that the following orbifolded LG theory appears:

$$\Big\{\widetilde{W}_{\ell} \;=\; \sum_{i=1}^{N} X_i^{\ell} + X_{P_2}^{-\frac{\ell}{N-\ell}} + \mathrm{e}^{t/\ell} X_1\cdots X_N X_{P_2}\Big\}\Big/(\mathbb{Z}_\ell)^N\;. \tag{3.13}$$

This theory is still ill-defined from the minimal model point of view. Even though the terms of positive powers such as $X_i^{\ell}$ are well-defined and they consist of $\mathcal{N}=2$ LG minimal model, there exists a term $X_{P_2}^{-\frac{\ell}{N-\ell}}$, which does not generate any critical points at finite $X_{P_2}$. However there is an interpretation to avoid this difficulty. Recall a discussion on the linear dilaton CFT and the Liouville theory [42, 43]. (We prepare a brief review in appendix C.) Based on this argument, we can interpret the negative power term corresponds to $Z_0^{-k}$ in (C.4), which gives an $\mathcal{N}=2$ SCFT on the coset $SL(2,\mathbb{R})_k/U(1)$ at level $k$ assigned by

$$k \;=\; \frac{\ell}{N-\ell}\;.$$

This assignment is correct because the conformal weights $r_a$ in the appendix C are all $r_a = 1/\ell$, where $n+1=N$. Thus we obtain $r_\Omega = \sum_a r_a - 1 = N/\ell - 1 \equiv 1/k$, which gives the above equation. This

[2] It is not serious to ignore an overall constant.

theory is given as an $\mathcal{N}=2$ Kazama-Suzuki model on the coset $SL(2,\mathbb{R})_k/U(1)$ at level $k$ [44], which is the gauged WZW model on the two-dimensional Euclidean black hole [21]. Furthermore this theory is exactly equivalent to $\mathcal{N}=2$ Liouville theory of background charge $Q^2=2/k$ via T-duality [43, 22]. We will continue to argue in later discussions.

### Solution two: $\mathbb{Z}_{N-\ell}$ orbifolded LG theory

In the same analogy of the previous discussion[3], we study the theory of the second solution

$$-Y_{P_2} \ = \ \frac{1}{N-\ell}\Big(t-\sum_{i=1}^{N}Y_i+\ell Y_{P_1}\Big) \ ,$$

which comes from the $\delta$-function in the period integral (3.10). Substituting this into (3.10), we find

$$\widehat{\Pi} \ = \ \int\prod_{i=1}^{N}\big(\mathrm{e}^{-\frac{1}{N-\ell}Y_i}\mathrm{d}Y_i\big)\,\big(\mathrm{e}^{\frac{\ell}{N-\ell}Y_{P_1}}\mathrm{d}Y_{P_1}\big)\,\exp\Big(-\sum_i\mathrm{e}^{-Y_i}-\mathrm{e}^{\frac{t}{N-\ell}}\prod_i\mathrm{e}^{-\frac{1}{N-\ell}Y_i}\mathrm{e}^{\frac{\ell}{N-\ell}Y_{P_1}}-\mathrm{e}^{-Y_{P_1}}\Big) \ .$$

Performing the re-definitions $X_i^{N-\ell}:=\mathrm{e}^{-Y_i}$ and $X_{P_1}^{N-\ell}:=\mathrm{e}^{\ell Y_{P_1}}$, we find that the period integral has a canonical measure and the "ill-defined" LG theory with orbifold symmetry appears:

$$\Big\{\widetilde{W}_{N-\ell} \ = \ \sum_{i=1}^{N}X_i^{N-\ell}+X_{P_1}^{-\frac{N-\ell}{\ell}}+\mathrm{e}^{\frac{t}{N-\ell}}X_1\cdots X_NX_{P_1}\Big\}\Big/(\mathbb{Z}_{N-\ell})^N \ . \tag{3.14}$$

Applying the discussions in appendix C to the negative power term in the superpotential $\widetilde{W}_{N-\ell}$, we find that the theory is also described by the well-defined LG theory with an orbifold symmetry coupled to $\mathcal{N}=2$ Kazama-Suzuki model on the coset $SL(2,\mathbb{R})_k/U(1)$ at level $k$, which is given by

$$\frac{N-\ell}{\ell} \ = \ k \ = \ \frac{2}{Q^2} \ .$$

where $Q$ is the charge of equivalent $\mathcal{N}=2$ Liouville theory.

## 3.4 Mirror geometry descriptions

In the previous subsection we found two orbifolded LG theories as exact effective theories. They are obtained by solving the twisted chiral superfields $Y_{P_1}$ and $Y_{P_2}$, respectively. Next we will read geometric informations from the same period integral (3.9b). Here we will also obtain two solutions which are related to the LG theories. The derivation procedure is so complicated that we try to imitate the method discussed in section 7.3 of [16] and we develop detailed calculations, explicitly. In order to obtain the geometric informations in the IR limit, we integrate out the superfield $\Sigma$ in the period integral (3.9b) after the replacement of $\ell\Sigma$ in (3.9b) to other variables, as we performed before.

[3]From now on we omit overall constant factors which appear in the period integral.

### $\mathbb{Z}_\ell$ orbifolded geometry

Let us study how to obtain the geometry with $\mathbb{Z}_\ell$-type orbifold symmetry. Replacing $\ell\Sigma$ in the period integral (3.9b) to

$$\ell\Sigma \ \to \ \frac{\partial}{\partial Y_{P_1}} \ ,$$

we can perform the integration of $\Sigma$ and obtain

$$\begin{aligned}\widehat{\Pi} \ &= \ \int \prod_{i=1}^{N} \mathrm{d}Y_i \big(\mathrm{e}^{-Y_{P_1}} \mathrm{d}Y_{P_1}\big) \mathrm{d}Y_{P_2} \\ &\qquad \times \delta\Big( \sum_i Y_i - \ell Y_{P_1} - (N-\ell) Y_{P_2} - t \Big) \exp\Big( - \sum_i \mathrm{e}^{-Y_i} - \mathrm{e}^{-Y_{P_1}} - \mathrm{e}^{-Y_{P_2}} \Big) \ . \end{aligned} \tag{3.15}$$

We perform the re-definitions of the variables $Y_i$, $Y_{P_1}$ and $Y_{P_2}$:

$$\begin{aligned} \mathrm{e}^{-Y_{P_1}} \ &=: \ \widetilde{P}_1 \ , & \mathrm{e}^{-Y_a} \ &=: \ \widetilde{P}_1 \, U_a & &\text{for } a = 1, \ldots, \ell \ , \\ \mathrm{e}^{-Y_{P_2}} \ &=: \ \widetilde{P}_2 \ , & \mathrm{e}^{-Y_b} \ &=: \ \widetilde{P}_2 \, U_b & &\text{for } b = \ell+1, \ldots, N \ . \end{aligned}$$

Substituting these re-defined variables into (3.15), we continue the calculation:

$$\begin{aligned} \widehat{\Pi} \ &= \ \int \prod_{i=1}^{N} \Big( \frac{\mathrm{d}U_i}{U_i} \Big) \mathrm{d}\widetilde{P}_1 \Big( \frac{\mathrm{d}\widetilde{P}_2}{\widetilde{P}_2} \Big) \delta\Big( \log\Big( \prod_i U_i \Big) + t \Big) \exp\Big\{ - \widetilde{P}_1 \Big( \sum_{a=1}^{\ell} U_a + 1 \Big) - \widetilde{P}_2 \Big( \sum_{b=\ell+1}^{N} U_b + 1 \Big) \Big\} \\ &= \ \int \prod_i \Big( \frac{\mathrm{d}U_i}{U_i} \Big) \mathrm{d}\widetilde{P}_2 \, \mathrm{d}u \, \mathrm{d}v \, \delta\Big( \log\Big( \prod_i U_i \Big) + t \Big) \, \delta\Big( \sum_a U_a + 1 \Big) \exp\Big\{ - \widetilde{P}_2 \Big( \sum_b U_b + 1 - uv \Big) \Big\} \\ &= \ \int \prod_i \Big( \frac{\mathrm{d}U_i}{U_i} \Big) \mathrm{d}u \, \mathrm{d}v \, \delta\Big( \log\Big( \prod_i U_i \Big) + t \Big) \, \delta\Big( \sum_a U_a + 1 \Big) \, \delta\Big( \sum_b U_b + 1 - uv \Big) \ , \end{aligned} \tag{3.16}$$

where we introduced new variables $u$ and $v$ taking values in $\mathbb{C}$ and used a following equation

$$\frac{1}{\widetilde{P}_2} \ = \ \int \mathrm{d}u \, \mathrm{d}v \, \exp\big( \widetilde{P}_2 \, uv \big) \ .$$

It is obvious that the resulting function (3.16) still includes a non-canonical integral measure. Thus we perform further re-definitions such as

$$U_a \ =: \ \mathrm{e}^{-t/\ell} \frac{Z_a^{\ell}}{Z_1 \cdots Z_N} \ , \qquad U_b \ =: \ Z_b^{\ell} \ .$$

Note that the period integral (3.16) is invariant under the following transformations acting on the new variables $Z_i$:

$$Z_a \ \mapsto \ \lambda \, \omega_a \, Z_a \ , \quad Z_b \ \mapsto \ \omega_b \, Z_b \ , \qquad \omega_a^{\ell} \ = \ \omega_b^{\ell} \ = \ \omega_1 \cdots \omega_N \ = \ 1 \ ,$$

where $\lambda$ is an arbitrary number taking in $\mathbb{C}^*$. The $\omega_i$ come from the shift symmetry of the original variables $Y_i \equiv Y_i + 2\pi i$. Combining these transformations we find that the period integral has $\mathbb{C}^* \times (\mathbb{Z}_\ell)^{N-2}$ symmetries. Substituting $Z_i$ into (3.16), we obtain

$$\widehat{\Pi} \ = \ \int \frac{1}{\text{vol.}(\mathbb{C}^*)} \prod_{i=1}^N \mathrm{d}Z_i \, \mathrm{d}u \, \mathrm{d}v \, \delta\Big( \sum_{a=1}^{\ell} Z_a^\ell + \mathrm{e}^{t/\ell} Z_1 \cdots Z_N \Big) \, \delta\Big( \sum_{b=\ell+1}^{N} Z_b^\ell + 1 - uv \Big) \ ,$$

which indicates that the resulting mirror geometry is described by

$$\widetilde{\mathcal{M}}_\ell \ = \ \Big\{ (Z_i; u, v) \in \mathbb{C}^{N+2} \, \Big| \, \big\{ \mathcal{F}(Z_i) \ = \ 0 \big\} / \mathbb{C}^* \ , \ \ \mathcal{G}(Z_b; u, v) \ = \ 0 \Big\} \Big/ (\mathbb{Z}_\ell)^{N-2} \ , \tag{3.17a}$$

$$\mathcal{F}(Z_i) \ := \ \sum_{a=1}^{\ell} Z_a^\ell + \psi Z_1 \cdots Z_\ell \ , \qquad \mathcal{G}(Z_b; u, v) \ := \ \sum_{b=\ell+1}^{N} Z_b^\ell + 1 - uv \ , \tag{3.17b}$$

$$\psi \ := \ \mathrm{e}^{t/\ell} Z_{\ell+1} \cdots Z_N \ . \tag{3.17c}$$

This is an $(N-1)$-dimensional complex manifold. It is guaranteed that $\widetilde{\mathcal{M}}_\ell$ is a CY manifold because of the following reason: We have already seen that the FI parameter $t$ in (3.9b) does not renormalized owing to the CY condition $\sum_a Q_a = 0$, which is also valid in the T-dual theory. In addition, we took the IR limit $e \to \infty$ and obtained the above non-trivial result. This means that the sigma model on the above geometry is a superconformal sigma model.

Let us study the manifold $\widetilde{\mathcal{M}}_\ell$ defined in (3.17) more in detail. The equation $\mathcal{F}(Z_i) = 0$ denotes that the complex variables $Z_a$ consist of the degree $\ell$ hypersurface in the projective space: $\mathbb{C}\mathbf{P}^{\ell-1}[\ell]$. This subspace itself is a compact CY manifold, which is parametrized by a parameter $\psi$ which is subject to the equation $\mathcal{G}(Z_b; u, v) = 0$. Moreover we can also interpret that the total space is a noncompact CY manifold whose compact directions are described by $Z_i$, while the variables $u$ and $v$ run in the noncompact directions under the equations (3.17b).

Here let us comment on a relation between the manifold $\widetilde{\mathcal{M}}_\ell$ and the LG twisted superpotential (3.13). As we have described in (3.17), $\widetilde{\mathcal{M}}_\ell$ has $(\mathbb{Z}_\ell)^{N-2}$ orbifold symmetry, while the LG theory (3.13) also holds this type of orbifold symmetry, i.e., the $(\mathbb{Z}_\ell)^N$ orbifold symmetry. When we combine the two equations in (3.17b) as follows:

$$F(Z_i, u, v) \ \equiv \ \mathcal{F}(Z_i) + \mathcal{G}(Z_b, u, v) \ = \ \sum_{i=1}^N Z_i^\ell + \mathrm{e}^{t/\ell} Z_1 \cdots Z_N + (1 - uv) \ = \ 0 \ .$$

This function $F = 0$ is quite similar to the LG twisted superpotential $\widetilde{W}_\ell$ including negative power term (3.13). Recall that a LG theory written by a superpotential $W$ is identical with a CY space defined by $W = 0$ in a (weighted) projective space. (See, for examples, [32, 35].) If we can apply this argument to the above result, the LG theory (3.13) is identical with the sigma model on (3.17) and there also exists the CY/LG correspondence in the T-dual theory.

### $\mathbb{Z}_{N-\ell}$ orbifolded geometry

We have constructed the two LG theories: $(\mathbb{Z}_\ell)^N$ orbifolded LG theory and $(\mathbb{Z}_{N-\ell})^N$ orbifolded LG theory. The former is related to the CY geometry $\widetilde{\mathcal{M}}_\ell$. It is natural to consider there also exists a dual geometry related to the latter LG theory. In the previous calculation, we replaced the $\ell\Sigma$ in the period integral (3.9b) to $\frac{\partial}{\partial Y_{P_1}}$ and we obtained the $(\mathbb{Z}_\ell)^{N-2}$ orbifolded geometry. Here we replace $\ell\Sigma$ to the differential with respect to $Y_{P_2}$, which is dual of the chiral superfield $P_2$ of charge $-(N-\ell)$:

$$\ell\Sigma \ \to \ \frac{\ell}{N-\ell}\frac{\partial}{\partial Y_{P_2}} \ .$$

Substituting this into (3.9b), we obtain the following expression:

$$\begin{aligned}\widehat{\Pi} \ &= \ \int \prod_{i=1}^N \mathrm{d}Y_i \, \mathrm{d}Y_{P_1} \big( \mathrm{e}^{-Y_{P_2}} \mathrm{d}Y_{P_2} \big) \\ &\qquad \times \delta\Big( \sum_i Y_i - \ell Y_{P_1} - (N-\ell) Y_{P_2} - t \Big) \exp\Big( - \sum_i \mathrm{e}^{-Y_i} - \mathrm{e}^{-Y_{P_1}} - \mathrm{e}^{-Y_{P_2}} \Big) \ . \end{aligned} \qquad (3.18)$$

Let us perform the following re-definitions of the variables $Y_i$, $Y_{P_1}$ and $Y_{P_2}$:

$$\begin{aligned} \mathrm{e}^{-Y_{P_1}} \ &=: \ \widetilde{P}_1 \ , \qquad & \mathrm{e}^{-Y_a} \ &=: \ \widetilde{P}_1 U_a \qquad & &\text{for } a = 1, \ldots, \ell \ , \\ \mathrm{e}^{-Y_{P_2}} \ &=: \ \widetilde{P}_2 \ , \qquad & \mathrm{e}^{-Y_b} \ &=: \ \widetilde{P}_2 U_b \qquad & &\text{for } b = \ell+1, \ldots, N \ . \end{aligned}$$

Substituting the re-defined variables into (3.18) and introducing auxiliary variables $u$ and $v$ in order to integrate out $\widetilde{P}_1$ completely, we obtain

$$\widehat{\Pi} \ = \ \int \prod_{i=1}^N \Big( \frac{\mathrm{d}U_i}{U_i} \Big) \mathrm{d}u \, \mathrm{d}v \, \delta\Big( \log \Big( \prod_{i=1}^N U_i \Big) + t \Big) \, \delta\Big( \sum_{b=\ell+1}^N U_b + 1 \Big) \, \delta\Big( \sum_{a=1}^{\ell} U_a + 1 - uv \Big) \ . \qquad (3.19)$$

The integral measure still remains non-canonical. We next introduce further re-definitions of $U_i$:

$$U_a \ =: \ Z_a^{N-\ell} \ , \qquad\qquad U_b \ =: \ \mathrm{e}^{-t/(N-\ell)} \frac{Z_b^{N-\ell}}{Z_1 \cdots Z_N} \ .$$

We can see that the map from $Z_i$ to $U_i$ is one-to-one modulo the $\mathbb{C}^* \times (\mathbb{Z}_{N-\ell})^{N-2}$ action given by

$$Z_a \ \mapsto \ \omega_a Z_a \ , \quad Z_b \ \mapsto \ \lambda \omega_b Z_b \ , \qquad \omega_a^{N-\ell} \ = \ \omega_b^{N-\ell} \ = \ \omega_1 \cdots \omega_N \ = \ 1 \ ,$$

where $\lambda$ takes value in $\mathbb{C}^*$. On account of the above re-definitions and symmetries we find that the period integral is re-written as

$$\widehat{\Pi} \ = \ \int \frac{1}{\mathrm{vol.}(\mathbb{C}^*)} \prod_{i=1}^N \mathrm{d}Z_i \, \mathrm{d}u \, \mathrm{d}v \, \delta\Big( \sum_{a=1}^{\ell} Z_a^{N-\ell} + 1 - uv \Big) \, \delta\Big( \sum_{b=\ell+1}^N Z_b^{N-\ell} + \mathrm{e}^{t/(N-\ell)} Z_1 \cdots Z_N \Big) \ ,$$

from which we can read the geometric information described by

$$\widetilde{\mathcal{M}}_{N-\ell} \ = \ \Big\{ (Z_i; u,v) \in \mathbb{C}^{N+2} \, \Big| \, \mathcal{F}(Z_a;u,v) \ = \ 0 \, , \ \ \{\mathcal{G}(Z_i) \ = \ 0\}/\mathbb{C}^* \Big\} \Big/ (\mathbb{Z}_{N-\ell})^{N-2} \, , \tag{3.20a}$$

$$\mathcal{F}(Z_a;u,v) \ := \ \sum_{a=1}^{\ell} Z_a^{N-\ell} + 1 - uv \, , \qquad \mathcal{G}(Z_i) \ := \ \sum_{b=\ell+1}^{N} Z_b^{N-\ell} + \psi Z_{\ell+1} \cdots Z_N \, , \tag{3.20b}$$

$$\psi \ := \ \mathrm{e}^{t/(N-\ell)} Z_1 \cdots Z_\ell \, . \tag{3.20c}$$

This is also a noncompact CY manifold including a compact CY hypersurface $\mathbb{C}\mathbf{P}^{N-\ell-1}[N-\ell]$, which is defined by $\mathcal{G}(Z_i) = 0$ and parametrized by $\psi$ with being subject to $\mathcal{F}(Z_a;u,v) = 0$. Since the variables are the twisted chiral superfields, we obtained the $\mathcal{N}=2$ supersymmetric NLSM on $\widetilde{\mathcal{M}}_{N-\ell}$ as a low energy effective theory of the T-dual theory. We can see that the sigma model on this manifold is identical with the LG theory described by (3.14).

### 3.5 Return to the gauged linear sigma model

As discussed before, it has been proved that the $\mathcal{N}=2$ SCFT on coset $SL(2,\mathbb{R})_k/U(1)$ at level $k$ is exactly T-dual with the $\mathcal{N}=2$ Liouville theory of background charge $Q$ under the relation $Q^2 = 2/k$. Let us apply this argument to the GLSM and its T-dual. Notice that the massless effective theories in the T-dual theory are exact, whereas the ones in the original GLSM are approximately realized.

Now let us recall that if a CFT $\mathcal{C}$ has an abelian discrete symmetry group $\Gamma$, the orbifold CFT $\mathcal{C}' = \mathcal{C}/\Gamma$ has a symmetry group $\Gamma'$ which is isomorphic to $\Gamma$ and a new orbifold CFT $\mathcal{C}'/\Gamma'$ is identical to the original CFT $\mathcal{C}$. Including this argument into the effective theories of the GLSM and its T-dual theory of $2 \leq \ell \leq N-1$, we find that the theories (2.15) and (2.16) are *equivalent* to (3.13) and (3.14), respectively. Furthermore we can interpret that the theories (2.15) and (2.16) are described by $\mathcal{N}=2$ Liouville theories coupled to the well-defined LG minimal models as *exact* effective theories. As a result we obtain the non-trivial relations among the various effective theories in the GLSM. Here we refer one typical result. The CY sigma model on (2.10) corresponds to (2.15), which is deformed to the LG theory coupled to the Liouville theory as an exact quantum theory. This is equivalent to (3.13) via T-duality. On account of the CY/LG correspondence, (3.13) and sigma model on (3.17) are identical with each other. Finally the original CY manifold (2.10) and (3.17) are mirror dual with each other. Notice that the CY manifold $\mathcal{M}_{\mathrm{CY}}$ is also deformed because the Liouville theory indicates that the dilaton field propagates on the target space [45, 46]. Of course we find that there are the same relations among effective theories (2.16), (2.17), (3.14) and (3.20).

Let us consider the case $\ell = 1$. As discussed before, the GLSM has only two massless effective theories (2.11) and (2.25). In addition, the subspace $\mathbb{C}\mathbf{P}^{\ell-1}[\ell]$ in (3.17) is ill-defined if $\ell = 1$ and then

the LG description (3.13) is also ill-defined. Thus the T-dual theory has only two descriptions (3.14) and (3.20) in the IR limit. This situation is consistent with the result in [16], where the GLSM for $\mathcal{O}(-N)$ bundle on $\mathbb{C}\mathbf{P}^{N-1}$ and its T-dual was discussed.

# 4 Summary and Discussions

We have studied the GLSM for noncompact CY manifolds realized as a line bundle on a hypersurface in a projective space. This gauge theory has three non-trivial phases and includes two types of four massless effective theories in the IR limit. Two theories are NLSMs on two distinct manifolds, whereas the other two are LG theories coupled to complex one-dimensional SCFTs. Following the conventional arguments, we have interpreted that these four theories are related to each other under phase transitions such as CY/LG correspondences and a topology change. Performing the T-duality, we have also obtained two types of four exact massless effective theories; the two theories are the sigma models on newly appeared mirror CY manifolds, while the other two are the LG theories including the terms of negative power $-k$, which may be regarded as indicating $\mathcal{N}=2$ SCFTs on coset $SL(2,\mathbb{R})_k/U(1)$ at level $k$. Since the SCFT on this coset is exactly equivalent to the $\mathcal{N}=2$ Liouville theory via T-duality, we have argued that the LG effective theories derived from the original GLSM are exactly realized by the Liouville theories coupled to the well-defined LG minimal models. The relations among the theories are illustrated in Figure 4:

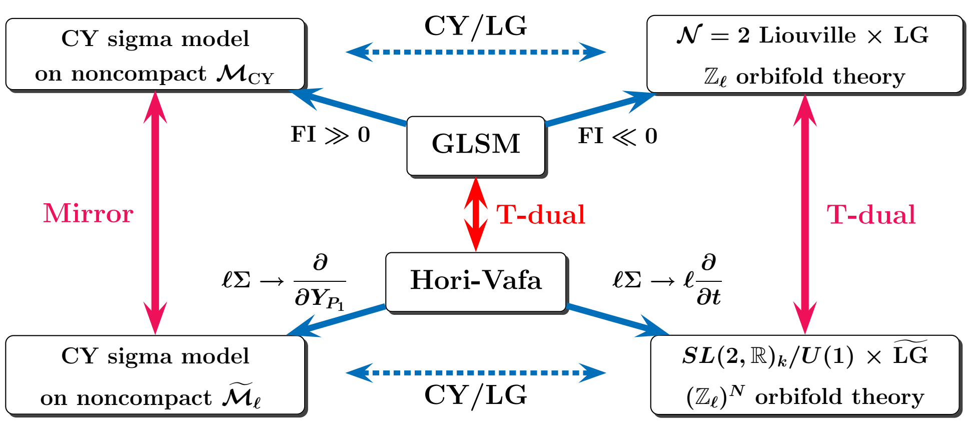


Figure 4: *Relations among IR effective theories of GLSM and its T-dual.*

Utilizing the above relations, we will obtain the topological charges of a CY manifold from the exact effective theories in the T-dual theory, even though we cannot directly calculate them in the original sigma model. Furthermore we will understand noncompact CY manifolds in detail from the

mathematical point of view. In addition, we can interpret the holographic duality in type II string theory on noncompact (singular) CY manifolds [43] as the phase transition and T-duality of the two-dimensional worldsheet theory and will be able to understand this duality more closely in the framework of the worldsheet sigma model description [47, 48, 49, 50].

As mentioned in the introduction, we have constructed the noncompact CY manifolds as line bundles on HSSs [7]. The base spaces HSSs can be seen as the submanifolds in the projective spaces obtained by polynomials with additional symmetries [6]: the quadric surface $SO(N)/[SO(N-2)\times U(1)]$ is given by a polynomial of degree two with $SO(N)$ symmetry, and $E_6/[SO(10)\times U(1)]$ has a set of differential equations including $E_6$ isometry group. These symmetries give the information of the complex structures of not only the base spaces HSSs but also the noncompact CY manifolds. However, the T-dual theory [16] is only valid when we consider the GLSM without a superpotential or with a superpotential given simply by a homogeneous polynomial such as $W_{\rm GLSM} = P \cdot G_{\ell}(S)$. Even though the polynomial $G_{\ell}(S)$ has an additional symmetry, the period integral (3.6) or (3.7) *cannot* recognize the existence of this additional symmetry. Thus the T-dual theory does not map all structures of the CY $\mathcal{M}$ to the mirror geometry completely. For example, we can argue the sigma model on the resolved conifold and its mirror dual in the framework of GLSM and its T-dual, however we have not even understood any correct descriptions for the deformed conifold represented by the GLSM. Therefore, if we wish to obtain the correct T-dual theories of the sigma models on such noncompact CY manifolds, we must improve the formulation so that it may recognize the complex structure of the manifold. It is quite significant to solve this problem in order to understand mirror symmetry for more general noncompact CY manifolds.

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# Appendix

## A Conventions

In this appendix we will write down the notation and convention which are modified from the ones defined in Wess-Bagger's book [51]. In [51] supersymmetric field theory is defined in four-dimensional spacetime. However in this paper we discuss supersymmetric field theory in two dimensions. Thus let us first perform the dimensional reduction. The coordinates in two dimensions $(x^0, x^1)$ are related to the four-dimensional ones $(y^0, y^1, y^2, y^3)$:

$$(x^0, x^1) \; \equiv \; (y^0, y^3) \, .$$

Note that we perform the dimensional reduction for $y^1$- and $y^2$-directions. Next we re-define the irreducible representation for spinors. Weyl spinors $\psi_\alpha$ in four dimensions becomes Dirac spinors. For convenience, we define the Dirac spinor indices in two dimensions as [14]:

$$\begin{gathered}(\psi^1, \psi^2) \; = \; (\psi^-, \psi^+) \, , \qquad (\psi_1, \psi_2) \; = \; (\psi_-, \psi_+) \, , \qquad \psi^- \; = \; \psi_+ \, , \qquad \psi^+ \; = \; -\psi_- \, , \\ \varepsilon^{12} \; = \; \varepsilon_{21} \; = \; 1 \quad \rightarrow \quad \varepsilon^{-+} \; = \; \varepsilon_{+-} \; = \; 1 \, , \\ (\psi^-, \psi^+)^\dagger \; = \; (\overline{\psi}{}^-, \overline{\psi}{}^+) \; = \; (\overline{\psi}_+, -\overline{\psi}_-) \, .\end{gathered}$$

Under the above convention, the super differential operators $D_\alpha$ are also changed as follows:

$$D_\pm \; = \; \frac{\partial}{\partial \theta^\pm} - i \overline{\theta}{}^\pm \big( \partial_0 \pm \partial_1 \big) \, , \qquad \overline{D}_\pm \; = \; - \frac{\partial}{\partial \overline{\theta}{}^\pm} + i \theta^\pm \big( \partial_0 \pm \partial_1 \big) \, .$$

Notice that the ordinary coordinate differentials are defined as $\partial_0 \equiv \partial / \partial x^0$ and $\partial_1 \equiv \partial / \partial x^1$. So far we wrote down the convention with respect to the two-dimensional Minkowski spacetime. When we consider a theory in two-dimensional Euclidean worldsheet, we modify the coordinate $x^0$ to $x^0 = -ix^2$.

Under the above convention, let us briefly introduce the following irreducible superfields, i.e., chiral superfields, vector (real) superfields and twisted chiral superfields.

**Chiral superfield**: As in the case of four-dimensional theory, a chiral superfield $\Phi$ is defined by $\overline{D}_\pm \Phi = 0$. We can expand a chiral superfield in terms of the fermionic coordinates $\{\theta^\pm, \overline{\theta}{}^\pm\}$ in the superspace:

$$\Phi(x, \theta, \overline{\theta}) \; = \; \phi(x) + \sqrt{2} \theta^+ \psi_+(x) + \sqrt{2} \theta^- \psi_-(x) + 2 \theta^+ \theta^- F(x) + \dots \; ,$$

where $\phi(x)$ is a complex scalar field, $\psi_\pm(x)$ are Dirac spinors and $F(x)$ is a complex auxiliary field, whose mass dimensions are 0, 1/2 and 1, respectively. The part written by "$+ \dots$" involves only the derivatives of these component fields $\phi$ and $\psi_\pm$.

**Vector superfield**: A vector superfield is defined by $V^\dagger = V$. This setup is also same in four-dimensional spacetime. We expand a vector superfield under the Wess-Zumino gauge:

$$\begin{aligned} V(x,\theta,\overline{\theta}) &= \theta^+\overline{\theta}^+\big(v_0+v_1\big) + \theta^-\overline{\theta}^-\big(v_0-v_1\big) - \sqrt{2}\theta^-\overline{\theta}^+\sigma - \sqrt{2}\theta^+\overline{\theta}^-\overline{\sigma} \\ &\quad - 2i\,\theta^+\theta^-\big(\overline{\theta}^+\overline{\lambda}_+ + \overline{\theta}^-\overline{\lambda}_-\big) + 2i\,\overline{\theta}^+\overline{\theta}^-\big(\theta^+\lambda_+ + \theta^-\lambda_-\big) - 2\theta^+\theta^-\overline{\theta}^+\overline{\theta}^- D\,. \end{aligned}$$

Note that we consider only $U(1)$ gauge theories in this paper, where $v_0$ and $v_1$ are components of a $U(1)$ gauge potential, $\lambda_\pm$ are gaugino fields as Dirac spinors and $D$ is a real auxiliary field. The complex fields $\sigma$ and $\overline{\sigma}$ are coming from the dimensionally reduced components of four-dimensional $U(1)$ gauge potential. In general we set this superfield to be dimensionless.

**Twisted chiral superfield**: A twisted chiral superfields is also an irreducible superfield in two dimensions. The definition is $\overline{D}_+ Y = D_- Y = 0$. Expanding a twisted chiral superfield $Y$ in terms of $\{\theta^\pm, \overline{\theta}^\pm\}$, we obtain

$$Y(x,\theta,\overline{\theta}) = y(x) + \sqrt{2}\theta^+\overline{\chi}_+(x) + \sqrt{2}\overline{\theta}^-\chi_-(x) + 2\theta^+\overline{\theta}^- G(x) + \ldots\,.$$

We denote a complex scalar field, Dirac spinors and an auxiliary field as $y(x)$, $\{\chi_-(x), \overline{\chi}_+(x)\}$ and $G(x)$, respectively. The part "$+\ldots$" means derivatives of component fields $y(x)$, $\chi_-(x)$ and $\overline{\chi}_+(x)$.

We can construct a superfield $\Sigma$ for the field strength $F_{mn} \equiv \partial_m v_n - \partial_n v_m$ in the following way:

$$\begin{aligned} \Sigma := \frac{1}{\sqrt{2}}\overline{D}_+ D_- V &= \sigma - i\sqrt{2}\,\theta^+\overline{\lambda}_+ - i\sqrt{2}\,\overline{\theta}^-\lambda_- + \sqrt{2}\,\theta^+\overline{\theta}^-(D - iF_{01}) \\ &\quad - i\overline{\theta}^-\theta^-\,(\partial_0-\partial_1)\,\sigma - i\theta^+\overline{\theta}^+\,(\partial_0+\partial_1)\,\sigma + \sqrt{2}\,\overline{\theta}^-\theta^+\theta^-\,(\partial_0-\partial_1)\,\overline{\lambda}_+ \\ &\quad + \sqrt{2}\,\theta^+\overline{\theta}^-\overline{\theta}^+\,(\partial_0+\partial_1)\,\lambda_- - \theta^+\overline{\theta}^-\theta^-\overline{\theta}^+\,({\partial_0}^2 - {\partial_1}^2)\,\sigma\,. \end{aligned}$$

This is also a twisted chiral superfield $\overline{D}_+\Sigma = D_-\Sigma = 0$. This superfield is gauge invariant under the $U(1)$ gauge transformation.

Here let us define integral measures of the fermionic coordinates $\theta^\pm$ and $\overline{\theta}^\pm$ in the superspace:

$$\mathrm{d}^2\theta := -\frac{1}{2}\,\mathrm{d}\theta^+\,\mathrm{d}\theta^-\,, \qquad \mathrm{d}^2\widetilde{\theta} := -\frac{1}{2}\,\mathrm{d}\theta^+\,\mathrm{d}\overline{\theta}^-\,, \qquad \mathrm{d}^4\theta := -\frac{1}{4}\mathrm{d}\theta^+\,\mathrm{d}\theta^-\,\mathrm{d}\overline{\theta}^+\,\mathrm{d}\overline{\theta}^-\,.$$

Thus the integral over $\theta^\pm$ and $\overline{\theta}^\pm$ are obtained as follows:

$$\int \mathrm{d}^2\theta\,\theta\theta = 1\,, \qquad \int \mathrm{d}^2\widetilde{\theta}\,\theta^+\overline{\theta}^- = \frac{1}{2}\,.$$

These definitions are slightly different from the ones in other papers, for example, [16, 17]. We notice that we use the above convention in this paper.

# B Weighted projective space

In this appendix we discuss a definition of one-dimensional weighted projective space. The weighted projective space is slightly different from the (ordinary) projective space. As we shall see, the most significant difference is that there exists an orbifold symmetry in the weighted projective space, while the projective space does not have this symmetry.

First let us review one-dimensional (ordinary) projective space $\mathbb{C}\mathbf{P}^1$. We prepare a two-dimensional complex plane without the origin such as $W = \mathbb{C}^2 - \{0\}$. The coordinates on this plane are described as $(z_1, z_2)$. The projective space $\mathbb{C}\mathbf{P}^1$ is defined as a space whose coordinate is given by the ratio of the complex variables $z_1$ and $z_2$. (The complex variables are called homogeneous coordinates in the projective space.) Under this definition, the two points $(z_1, z_2)$ and $(\lambda z_1, \lambda z_2)$ in $W$-plane are identified with each other:

$$(z_1, z_2) \ \simeq \ (\lambda z_1, \lambda z_2) \ , \tag{B.1}$$

where $\lambda$ is a variable of $\mathbb{C}^* = \mathbb{C} - \{0\}$. In other words, all the points on the straight line through the origin of $\mathbb{C}^2$ are identified via the above projection. Note that the projective space $\mathbb{C}\mathbf{P}^1$ is diffeomorphic to the two-sphere: $\mathbb{C}\mathbf{P}^1 \simeq S^2$.

Next we define a weighted projective space $\mathbf{W}\mathbb{C}\mathbf{P}^1_{\ell, N-\ell}$. Here we also prepare a two-dimensional complex plane $W$, whose coordinates are expressed by $(z_1, z_2)$. The weighted projective space is given as a space of complex coordinate defined by the ratio of $z_1$ and $z_2$ with appropriate weights. The identification in the $W$-plane is the following:

$$(z_1, z_2) \ \simeq \ (\lambda^{\ell} z_1, \lambda^{N-\ell} z_2) \ , \tag{B.2}$$

where both $\ell$ and $N - \ell$ are positive integers: $\ell, N - \ell \in \mathbb{Z}_{>0}$. This identification has a residual symmetry with respect to the phases such as

$$(\omega^{\ell} z_1, \omega^{N-\ell} z_2) \ = \ (z_1, z_2) \ , \tag{B.3}$$

where $\omega = \exp(2\pi i\, \alpha)$ is the phase of $\lambda$ and $\alpha$ is a great common number of $\ell$ and $N - \ell$ described by $\alpha = \mathrm{GCM}\{\ell, N - \ell\}$. This does not exist in the definition of $\mathbb{C}\mathbf{P}^1$. In the case of $\mathbb{C}\mathbf{P}^1$, the identification (B.1) fixes the phase of homogeneous coordinates $z_1$ and $z_2$ completely. On the other hand, the identification (B.2) does not fix the phases of $z_1$ and $z_2$, and the residual symmetry (B.3) exists. Due to this, roughly speaking, we can see that the weighted projective space is a projective space with $\mathbb{Z}_{\alpha}$ orbifold symmetry. There are two specific points. On the point $(z_1, z_2) = (z_1, 0)$, the orbifold symmetry is enhanced to $\mathbb{Z}_{\ell}$, the other point $(z_1, z_2) = (0, z_2)$ generates $\mathbb{Z}_{N-\ell}$. In addition, if

we choose $\ell = N - \ell$, the weighted projective space can be reduced to the ordinary projective space $\mathbf{W}\mathbb{C}\mathbf{P}^1_{\ell,N-\ell=\ell} = \mathbb{C}\mathbf{P}^1$, which has no longer an orbifold symmetry.

# C Linear dilaton CFT and Liouville theory

In this appendix we demonstrate the linear dilaton CFT and the Liouville theory discussed in [42, 43]. Let us consider the superstring propagating on the following the ten-dimensional spacetime:

$$\mathbb{R}^{d-1,1} \times X^{2n} , \qquad 2n \ = \ 10 - d ,$$

where $X^{2n}$ is a $2n$-dimensional singular CY manifold. Sending the zero string coupling limit $g_s \to 0$ at fixed string length $l_s$ gives rise to a $d$-dimensional theory without gravity describing the dynamics of modes living near the singularity on $X^{2n}$. This theory is holographic dual to string theory on a following background which approaches at weak coupling region:

$$\mathbb{R}^{d-1,1} \times \mathbb{R}_\phi \times \mathcal{M} \ = \ \mathbb{R}^{d-1,1} \times \mathbb{R}_\phi \times S^1 \times \mathcal{M}/U(1) ,$$

where $\mathcal{M}$ is a compact and non-singular manifold. The real line $\mathbb{R}_\phi$ is parameterized by $\phi$.

We can define CFT on each subspace. On the flat space $\mathbb{R}^{d-1,1}$ we can define $\mathcal{N}=1$ SCFT whose central charge is

$$c_d \ = \ \frac{3}{2} d . \tag{C.1}$$

We describe the theory on $\mathbb{R}_\phi$ in terms of a linear dilaton given by $\Phi = -\frac{Q}{2}\phi$. The linear dilaton CFT has a central charge

$$c_\phi \ = \ 1 + 3Q^2 . \tag{C.2}$$

From the consistency of superstring propagation, the worldsheet theory on $\mathcal{M}$ should be an $\mathcal{N}=1$ SCFT with central charge $c_{\mathcal{M}} = 3(n - 1/2 - Q^2)$. Moreover, if the manifold has a $U(1)$ symmetry, the theory on the coset manifold $\mathcal{M}/U(1)$ must be an extended $\mathcal{N}=2$ SCFT with central charge

$$c_{\mathcal{M}/U(1)} \ = \ 3(n - 1 - Q^2) . \tag{C.3}$$

Let us specialize the $\mathcal{N}=2$ SCFT on $\mathcal{M}/U(1)$ to the $\mathcal{N}=2$ LG minimal model whose superpotential is defined in terms of $n+1$ chiral superfields $Z_a$:

$$W_{\rm LG} \ = \ F(Z_a) , \qquad F(\lambda^{r_a} Z_a) \ = \ \lambda F(Z_a) , \qquad a = 1,2,\ldots,n+1 ,$$

where $r_a$ are the conformal weights of the chiral superfields $Z_a$, respectively. Note that we have already understood properties of this minimal model [15, 31]. The worldsheet central charge should correspond to (C.3) such as

$$c_{\rm LG} \ = \ 3 \sum_{a=1}^{n+1} \big( 1 - 2 r_a \big) \ \equiv \ c_{\mathcal{M}/U(1)} \ .$$

If we introduce a new variable $r_{\Omega}$ with respect to the conformal weights $r_a$ such as $r_{\Omega} \equiv \sum_a r_a - 1$, we can express the background charge $Q$ to $Q^2 = 2 r_{\Omega}$.

Here let us combine the above discussion with the conjectures proposed by Muhki and Vafa [52], Ghoshal and Vafa [53], and Ooguri and Vafa [54], where they insisted that an $\mathcal{N}=2$ SCFT on the noncompact space $\mathbb{R}_{\phi} \times \mathcal{M}$ can be given formally by the "LG" superpotential

$$W \ = \ - \mu Z_0^{-k} + F(Z_a) \ , \tag{C.4}$$

where $Z_0$ is an additional chiral superfield and

$$k \ = \ \frac{1}{r_{\Omega}} \ = \ \frac{2}{Q^2} \ . \tag{C.5}$$

This formulation is useful to describe the sigma model on deformed conifold [53]. The first term in the superpotential appears to be ill-defined from the LG minimal model point of view. The corresponding potential does not have a minimum at the finite value of $Z_0$. The topological LG model with such a superpotential has already been studied by Ghoshal and Mukhi [55], and Hanany, Oz and Ronen Plesser [56] in order to investigate two-dimensional string theory. Moreover, in general, $k$ is not an integer, which makes (C.4) non-single valued. Thus, it was proposed that this first term can be interpreted as an $\mathcal{N}=2$ SCFT on the coset $SL(2,\mathbb{R})/U(1)$ at level $k$. From the geometric point of view, this coset space corresponds to a semi-infinite cigar, and in the IR limit this geometry deforms to the two-dimensional Euclidean black hole [21]. This SCFT on the coset had been believed to be isomorphic to the Liouville theory in the sense of SCFT. They are related by strong-weak coupling duality on the worldsheet: The theory (C.4) can be valid as a Liouville theory in the large $Q$ limit, while this can be seen as a coset SCFT in the large $k$ limit ($k = 2/Q^2$). Finally, it has been proved that the $\mathcal{N}=2$ SCFT on the coset $SL(2,\mathbb{R})_k/U(1)$ is exactly equivalent (or T-dual) to the $\mathcal{N}=2$ Liouville theory to each other in any values of $k > 0$ [22]. This equivalence was also proved by Tong in the framework of two-dimensional domain wall physics in three-dimensional theory [57].

To summarize, we find that the string theory on a singular CY manifold $X^{2n}$ can be holographic dual to string theory as a product theory of the $\mathcal{N}=2$ SCFT on the coset $SL(2,\mathbb{R})_k/U(1)$ and the $\mathcal{N}=2$ LG minimal model on $\mathcal{M}/U(1)$. The coset SCFT sector is also equivalent to the $\mathcal{N}=2$ Liouville theory on $\mathbb{R}_{\phi} \times S^1$.

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